On the structure and clique-width of $(4K_1, C_4, C_6, C_7)$-free graphs

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Abstract

We give a complete structural description of $(4K_1, C_4, C_6, C_7)$-free graphs that do not contain a simplicial vertex, and we prove that such graphs have bounded clique-width. Together with the results of Foley *et al* [*Graphs and Combinatorics*, 36:125–138, 2020], this implies that that $(4K_1, C_4, C_6)$-free graphs that do not contain a simplicial vertex have bounded clique-width. Consequently, Graph Coloring can be solved in polynomial time for $(4K_1, C_4, C_6)$-free graphs, *i.e.* for even-hole-free graphs of stability number at most three.

1 Introduction

All graphs in this paper are finite, simple, and nonnull.

As usual, for a positive integer k, K_k is the complete graph on k vertices, and C_k (for $k \geq 3$) is the cycle on k vertices. For a positive integer k and a graph H, we denote by kH the disjoint union of k copies of H; in particular, $4K_1$ is the edgeless graph on four vertices. For a graph H, a graph G is said to be *H-free* if no induced subgraph of G is isomorphic to H. For a family of graphs $\mathcal{H}$, a graph G is said to be *$\mathcal{H}$-free* if G is H-free for all $H \in \mathcal{H}$. A *hole* in a graph G is an induced cycle of length at least four in G. A hole is *even* or *odd* depending on the parity of its length.

A *clique* in a graph G is a (possibly empty) set of pairwise adjacent vertices, and a *stable set* in G is a (possibly empty) set of pairwise nonadjacent vertices. The *clique number* of G, denoted by $\omega(G)$, is the maximum size of a clique in G; the *stability number* of G, denoted by $\alpha(G)$, is the maximum size of a stable set in G. Note that a graph G is $4K_1$-free if and only if $\alpha(G) \leq 3$.

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A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. For an integer k, a graph G is said to be *k-colorable* if there exists a proper coloring of G that uses at most k colors. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest nonnegative integer k such that G is k-colorable. Graph Coloring is the following problem.

Graph Coloring
Instance: A graph G and a nonnegative integer k.
Question: Is G k-colorable?

Graph Coloring is NP-hard in general, but it becomes solvable in polynomial time when restricted to certain classes of graphs. In this context, the class of even-hole-free graphs is of particular interest. This is a well-studied class: there are decomposition theorems [6, 8], as well as polynomial-time recognition algorithms [4, 7] for it. Furthermore, every even-hole-free graph contains a "bisimplicial vertex" (*i.e.* a vertex whose neighborhood is the union of two cliques) [5]; this readily implies that every even-hole-free graph G satisfies $\chi(G) \leq 2\omega(G) - 1$, *i.e.* the class of even-hole-free graphs is "χ-bounded" by a linear function. The Maximum Clique problem is solvable in polynomial time for C_4-free graphs [1, 9, 13, 15],[1] and therefore for even-hole-free graphs as well. However, the complexity of the Graph Coloring problem (as well as of the Maximum Stable Set problem) is still open for this class. The complexity of Graph Coloring is also open for the class of $(4K_1, C_4)$-free graphs. Foley *et al* [10] raised (and partially answered; see below) the question of whether Graph Coloring is solvable in polynomial time for the intersection of these two classes, *i.e.* for the class of even-hole-free graphs of stability number at most three. Since every cycle of length at least eight contains a stable set of size four, even-hole-free graphs of stability number at most three are precisely the $(4K_1, C_4, C_6)$-free graphs. Note that graphs in this class are trivially recognizable in $O(n^6)$ time. Note, furthermore, that all holes in an $(4K_1, C_4, C_6)$-free graph are of length five or seven.

The *clique-width* of a graph G, denoted by $\mathrm{cwd}(G)$, is the minimum number of labels needed to construct G using the following four operations:

1. creation of a new vertex v with label i;
2. disjoint union of two labeled graphs;

[1]Indeed, any C_4-free graph has only $O(n^2)$ maximal cliques [1, 9], and if a graph G has K maximal cliques, they can all be found in $O(Kn^3)$ time by combining results from [13, 15]. So, all maximal cliques of a C_4-free graph can be found in $O(n^5)$ time, and then a maximum clique can be found by comparing the size of all maximal cliques.

3. joining by an edge every vertex labeled i to every vertex labeled j (where $i \neq j$);

4. renaming label i to label j.

Theorem 1.1. *[14] The* Graph Coloring *problem can be solved in polynomial time for graphs of bounded clique-width.*

Foley *et al* [10] gave a full structural description of $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7. As an easy corollary, they obtained the following theorem.

Theorem 1.2. *[10]* $(4K_1, C_4, C_6)$*-free graphs that contain an induced* C_7 *have bounded clique-width.*

Clearly, Theorems 1.1 and 1.2 together imply that Graph Coloring can be solved in polynomial time for $(4K_1, C_4, C_6)$-free graphs that contain an induced C_7. In view of these results, Foley *et al* [10] asked whether Graph Coloring can be solved in polynomial time for $(4K_1, C_4, C_6, C_7)$-free graphs.

A *simplicial vertex* is a vertex whose neighborhood is a (possibly empty) clique. Our main result is a decomposition theorem for $(4K_1, C_4, C_6, C_7)$-free graphs that do not contain a simplicial vertex; more precisely, we give a full structural description of $(4K_1, C_4, C_6, C_7)$-free graphs that do not contain a simplicial vertex (see Theorem 2.2). Using our structural results, we prove the following.

Theorem 1.3. *Let* G *be a* $(4K_1, C_4, C_6, C_7)$*-free graph. Then either* G *has a simplicial vertex, or* G *satisfies* $cwd(G) \leq 5$.

We prove Theorem 1.3 in Section 3. A graph is *chordal* if it contains no holes. It is well-known that every chordal graph contains a simplicial vertex [11]. Clearly, every $4K_1$-free chordal graph is $(4K_1, C_4, C_6)$-free, and it was shown in [2] that $4K_1$-free chordal graphs have unbounded clique-width. Thus, the "simplicial vertex" outcome cannot be removed from Theorem 1.3, even if the bound on clique-width is increased.

Theorems 1.1, 1.2, and 1.3 together imply that Graph Coloring can be solved in polynomial time for $(4K_1, C_4, C_6)$-free graphs, *i.e.* for even-hole-free graphs of stability at most three (see Corollary 1.4 below). The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is the number of neighbors of v in G. Note that if v is a simplicial vertex of G, then $d_G(v) \leq \omega(G) - 1 \leq \chi(G) - 1$.

Corollary 1.4. *The* Graph Coloring *problem can be solved in polynomial time for* $(4K_1, C_4, C_6)$*-free graphs.*

Proof (assuming Theorem 1.3). Clearly, there is an $O(n^3)$ time algorithm that either finds a simplicial vertex in an arbitrary input graph, or determines that the graph has no simplicial vertices (we simply examine the neighborhood of each vertex). Furthermore, if v is a simplicial vertex of a graph G on at least two vertices, then $\chi(G) = \max\{d_G(v) + 1, \chi(G \setminus v)\}$, and so G is k-colorable if and only if $d_G(v) \leq k - 1$ and $G \setminus v$ is k-colorable. On the other hand, Theorems 1.2 and 1.3 guarantee that $(4K_1, C_4, C_6)$-free graphs that contain no simplicial vertices have bounded clique-width, and by Theorem 1.1, Graph Coloring can be solved in polynomial time for such graphs. □

The remainder of this paper is organized as follows. In Section 1.1, we introduce some (mostly standard) terminology and notation, which we use throughout the paper, and we also prove a few simple lemmas. In Section 2, we state and prove Theorem 2.2, which is our structure theorem for $(4K_1, C_4, C_6, C_7)$-free graphs that do not contain a simplicial vertex. In Section 3, we prove Theorem 1.3.

1.1 Terminology and notation (and some easy lemmas)

As usual, the vertex and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex x in a graph G, the *open neighborhood* (or simply *neighborhood*) of x in G, denoted by $N_G(x)$, is the set of all neighbors of x in G, and the *closed neighborhood* of x in G, denoted by $N_G[x]$, is defined as $N_G[x] = \{x\} \cup N_G(x)$. Recall that the *degree* of x in G, denoted by $d_G(x)$, is the number of neighbors that x has in G, *i.e.* $d_G(x) = |N_G(x)|$.

Given a graph G and distinct vertices $x, y \in V(G)$, we say that x *dominates* y in G, or that y is *dominated* by x in G, provided that $N_G[y] \subseteq N_G[x]$.[2] A vertex $v \in V(G)$ is *universal* in G if v is adjacent to all other vertices of G, *i.e.* if $N_G[v] = V(G)$.

For a graph G and a nonempty set $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S; for vertices $x_1, \dots, x_t \in V(G)$, we sometimes write $G[x_1, \dots, x_t]$ instead of $G[\{x_1, \dots, x_t\}]$. For a set $S \subsetneqq V(G)$, $G \setminus S$ is the subgraph of G obtained by deleting S, *i.e.* $G \setminus S = G[V(G) \setminus S]$. If G has at least two vertices and $x \in V(G)$, we sometimes write $G \setminus x$ instead of $G \setminus \{x\}$.[3]

For an integer $k \geq 4$, a *k-hole* in a graph G is an induced C_k in G. When we write "$x_1, \dots, x_k, x_1$ is a k-hole in G," where $k \geq 4$, we mean that $x_1, \dots, x_k$ are pairwise distinct vertices of G, and furthermore, the edges of $G[x_1, \dots, x_k]$ are precisely $x_1x_2, x_2x_3, x_3x_4, \dots, x_{k-1}x_k, x_kx_1$.

Given a graph G, a vertex $x \in V(G)$, and a set $Y \subseteq V(G) \setminus \{x\}$, we say that x is *complete* (resp. *anticomplete*) to Y in G provided that x is adjacent

[2] Note that this implies that x and y are adjacent.

[3] Since our graphs are nonnull, if $V(G) = \{x\}$, then $G \setminus x$ is not defined.

(resp. nonadjacent) to all vertices of Y in G.

Given a graph G and disjoint sets $X, Y \subseteq V(G)$, we say that X is *complete* (resp. *anticomplete*) to Y in G provided that every vertex in X is complete to Y in G.

As usual, the *complement* of a graph G, denoted by $\overline{G}$, is the graph whose vertex set is $V(G)$ and in which two distinct vertices are adjacent if and only if they are nonadjacent in G. A graph is *anticonnected* if its complement is connected. Obviously, every anticonnected graph on at least two vertices contains a pair of nonadjacent vertices.

An *anticomponent* of a graph G is an induced subgraph Q of G such that $\overline{Q}$ is a connected component of $\overline{G}$. An anticomponent is *trivial* if it has only one vertex, and it is *nontrivial* if it has at least two vertices. Clearly, the vertex sets of the anticomponents of a graph G are complete to each other in G.[4]

Lemma 1.5. *If a graph is C_4-free, then it has at most one nontrivial anticomponent. Furthermore, if a C_4-free graph contains no simplicial vertices, then it has exactly one nontrivial anticomponent.*

Proof. If a graph contains no simplicial vertices, then it is not complete, and consequently, it has at least one nontrivial anticomponent.

It remains to show that any graph with at least two nontrivial anticomponents contains a 4-hole. So, let G be a graph, and suppose that X, Y are the vertex sets of two distinct, nontrivial anticomponents of G. Then X and Y are complete to each other. Since $G[X]$ is anticonnected and $|X| \geq 2$, we see that there exist distinct, nonadjacent vertices $x_1, x_2 \in X$. Similarly, there exist distinct, nonadjacent vertices $y_1, y_2 \in Y$. But now x_1, y_1, x_2, y_2, x_1 is a 4-hole in G. □

Lemma 1.6. *Let G be a graph that has exactly one nontrivial anticomponent, call it Q. Then both the following hold:*

(a) G has a simplicial vertex if and only if Q has a simplicial vertex;

(b) if H is a graph that contains no universal vertices, then G is H-free if and only if Q is H-free.

Proof. We first prove (a). We remark that $V(G) \setminus V(Q)$ is a (possibly empty) clique, complete to $V(Q)$ in G. Since Q contains a pair of nonadjacent vertices, it follows that no vertex in $V(G) \setminus V(Q)$ is simplicial in G. On the other hand, for every vertex $v \in V(Q)$, we have that $N_G(v) = N_Q(v) \cup$

[4]Thus, when we say that "Q is the only nontrivial anticomponent of G," we have that Q is an anticonnected induced subgraph of G, that $|V(Q)| \geq 2$, and that $V(G) \setminus V(Q)$ is a (possibly empty) clique, complete to $V(Q)$ in G. In particular, either $G = Q$, or G can be obtained from Q by repeatedly adding universal vertices.

$(V(G) \setminus V(Q))$, and we deduce that v is simplicial in G if and only if it is simplicial in Q. This proves (a).

It remains to prove (b). Fix a graph H that has no universal vertices. If G is H-free, then it is clear that Q is H-free. Suppose now that G is not H-free, and fix some $X \subseteq V(G)$ such that $G[X]$ is isomorphic to H. Suppose that $X \not\subseteq V(Q)$, and fix some $x \in X \setminus V(Q)$. Then $x \in V(G) \setminus V(Q)$; consequently, x is a universal vertex of G, and therefore of $G[X]$ as well. But this is impossible, since $G[X]$ is isomorphic to H, and H has no universal vertices. So, $X \subseteq V(Q)$. Then $Q[X]$ is isomorphic to H, and so Q is not H-free. This proves (b). □

A *cutset* of a graph G is a (possibly empty) set $C \subsetneqq V(G)$ such that $G \setminus C$ is disconnected. A *cut-partition* of a graph G is a partition (A, B, C) of $V(G)$ such that A and B are nonempty and anticomplete to each other in G (the set C may possibly be empty). Clearly, if (A, B, C) is a cut-partition of G, then C is a cutset of G. Conversely, every cutset of G gives rise to at least one cut-partition of G.

A *clique-cutset* of a graph G is a cutset of G that is also a clique of G. (Note that if G is disconnected, then $\emptyset$ is a clique-cutset of G.) A *clique-cut-partition* of a graph G is a cut-partition (A, B, C) of G such that C is a (possibly empty) clique of G. Clearly, if (A, B, C) is a clique-cut-partition of G, then C is a clique-cutset of G. Conversely, every clique-cutset of G gives rise to at least one clique-cut-partition of G.

Lemma 1.7. *Every $(4K_1, C_4)$-free graph that admits a clique-cutset contains a simplicial vertex. More precisely, for every $(4K_1, C_4)$-free graph G, and every clique-cut-partition (A, B, C) of G, all the following hold:*

(a) at least one of A and B is a clique;

(b) if A is a clique and $a \in A$ is chosen so that $d_G(a)$ is as small as possible, then a is simplicial in G;[5]

(c) if B is a clique and $b \in B$ is chosen so that $d_G(b)$ is as small as possible, then b is simplicial in G.[6]

Proof. Let G be a $(4K_1, C_4)$-free graph, and let (A, B, C) be a clique-cut-partition of G.

If (a) is false, then we choose distinct, nonadjacent vertices $a_1, a_2 \in A$, we choose distinct, nonadjacent vertices $b_1, b_2 \in B$, and we observe that $\{a_1, a_2, b_1, b_2\}$ is a stable set of size four in G, contrary to the fact that G is $4K_1$-free. So, (a) holds.

[5] More precisely, we assume that $a \in A$ satisfies the property that for all $a' \in A$, we have that $d_G(a) \leq d_G(a')$.

[6] More precisely, we assume that $b \in B$ satisfies the property that for all $b' \in B$, we have that $d_G(b) \leq d_G(b')$.

We now prove (b). Suppose that A is a clique, and choose $a \in A$ such that $d_G(a)$ is as small as possible. We claim that a is simplicial in G. Suppose otherwise, and fix distinct, nonadjacent vertices $x, y \in N_G(a)$. Clearly, $N_G(a) \subseteq A \cup C$; since A and C are cliques, we deduce that one of x, y belongs to A, and the other one belongs to C. By symmetry, we may assume that $x \in A$ and $y \in C$. Since y is a neighbor of a but not of x, the minimality of $d_G(a)$ implies that there is a vertex $z \in V(G) \setminus \{a, x\}$ that is adjacent to x, but not to a. Since z is nonadjacent to $a \in A$, and since A is a clique, it follows that $z \notin A$; since $N_G(x) \subseteq A \cup C$, we deduce that $z \in C$. Since $y, z \in C$, and since C is a clique, we see that y, z are adjacent. But now a, x, z, y, a is a 4-hole in G, contrary to the fact that G is C_4-free. This proves (b). The proof of (c) is analogous. □

2 Decomposing $(4K_1, C_4, C_6, C_7)$-free graphs

In this section, we state and prove a decomposition theorem for the class of $(4K_1, C_4, C_6, C_7)$-free graphs. More precisely, we give a full structural description of $(4K_1, C_4, C_6, C_7)$-free graphs that do not contain a simplicial vertex (see Theorem 2.2). We begin by defining "5-baskets," "rings," and "5-crowns"; these graphs[7] appear in the statement of Theorem 2.2.

A *5-basket* is a graph Q whose vertex set can be partitioned into sets $A, B_1, B_2, B_3, C_1, C_2, C_3, F$ such that all the following hold:

- $A, B_1, B_2, B_3, C_1, C_2, C_3$ are nonempty cliques;
- F is a (possibly empty) clique;
- cliques B_1, B_2, B_3 are pairwise anticomplete to each other;
- cliques C_1, C_2, C_3 are pairwise complete to each other;
- there exists an index $i^* \in \{1, 2, 3\}$ such that
  - A is complete to $(B_1 \cup B_2 \cup B_3) \setminus B_{i^*}$, and
  - A can be ordered as $A = \{a_1, \dots, a_t\}$ so that $N_Q(a_t) \cap B_{i^*} \subseteq \dots \subseteq N_Q(a_1) \cap B_{i^*} = B_{i^*}$;[8]
- A is anticomplete to $C_1 \cup C_2 \cup C_3$;
- for all indices $i \in \{1, 2, 3\}$, B_i is complete to C_i and anticomplete to $(C_1 \cup C_2 \cup C_3) \setminus C_i$;
- there exists an index $j^* \in \{1, 2, 3\}$ such that F is complete to $V(Q) \setminus (B_{j^*} \cup C_{j^*} \cup F)$ and anticomplete to $B_{j^*} \cup C_{j^*}$.

[7] In fact, only 5-baskets and 5-crowns.

[8] Thus, a_1 is complete to $B_1 \cup B_2 \cup B_3$. Furthermore, B_{i^*} can be ordered as $B_{i^*} = \{b_1, \dots, b_p\}$ so that $a_1 \in N_Q(b_p) \cap A \subseteq \dots \subseteq N_Q(b_1) \cap A$.

Under such circumstances, we say that $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ is a *5-basket partition* of the 5-basket Q.

Note that there are effectively two different types of 5-basket (depending on whether or not i^* and j^* are the same). These two types of 5-basket (up to a permutation of the index set $\{1, 2, 3\}$) are represented in Figure 1.

A *ring* (originally introduced in [3] and further studied in [12]) is a graph R whose vertex set can be partitioned into $k \geq 4$ nonempty sets, say $X_0, \dots, X_{k-1}$ (with indices understood to be in $\mathbb{Z}_k$), such that for all $i \in \mathbb{Z}_k$, X_i can be ordered as $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$. (Note that this implies that $X_0, \dots, X_{k-1}$ are all cliques.) Under these circumstances, we also say that the ring R is of *length* k, as well as that R is a k*-ring*. A ring is *long* if it is of length at least five. Furthermore, we say that $(X_0, \dots, X_{k-1})$ is a *ring partition* of the ring R.

A *5-crown* is a 5-ring R with ring partition $(X_0, X_1, X_2, X_3, X_4)$ such that for some index $i^* \in \mathbb{Z}_5$, we have that X_{i^*-1} is complete to X_{i^*-2}, and X_{i^*+1} is complete to X_{i^*+2}. Under such circumstances, we say that $(X_0, X_1, X_2, X_3, X_4)$ is a *5-crown partition* of the 5-crown R. A 5-crown with $i^* = 0$ is represented in Figure 2.

Lemma 2.1. *5-Baskets and 5-crowns are anticonnected and do not contain simplicial vertices.*

Proof. This readily follows from the relevant definitions. □

We are now ready to state Theorem 2.2, the main result of this section.

Theorem 2.2. *Let G be a graph. Then the following two statements are equivalent:*

- *G is a $(4K_1, C_4, C_6, C_7)$-free graph that does not contain a simplicial vertex;*
- *G has exactly one nontrivial anticomponent, and this anticomponent is either a 5-basket or a 5-crown.*

By Lemma 2.1, 5-baskets and 5-crowns are anticonnected and contain no simplicial vertices. So, Theorem 2.2 implies that $(4K_1, C_4, C_6, C_7)$-free graphs that contain no simplicial vertices are precisely those graphs that can be obtained from a 5-basket or 5-crown by (possibly) repeatedly adding universal vertices. Note, however, that adding simplicial vertices can possibly introduce an induced $4K_1$, and so Theorem 2.2 is not quite a structure theorem for the class of $(4K_1, C_4, C_6, C_7)$-free graphs.

The remainder of this section is devoted to proving Theorem 2.2.

The *5-pyramid* is the seven-vertex graph represented in Figure 3. Note that the 5-pyramid has exactly three holes, and they are all of length five.

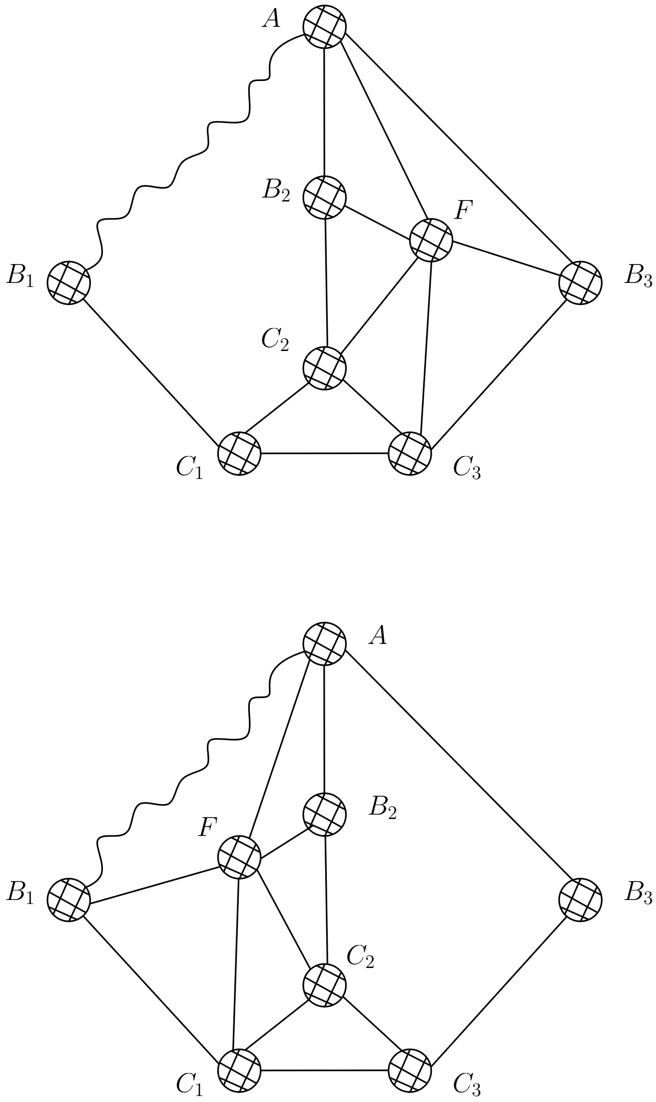

Figure 1: A 5-basket with 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$, and with $i^* = j^* = 1$ (top) or $i^* = 1$ and $j^* = 3$ (bottom). Crosshatched disks represent cliques (F may possibly be empty, and the other seven crosshatched disks represent nonempty cliques). A straight line between two disks indicates that the corresponding cliques are complete to each other. A wavy line between two disks indicates that there are edges between the corresponding cliques (those edges must obey the axioms from the definition of a 5-basket). The absence of a line (straight or wavy) between two disks indicates that the corresponding cliques are anticomplete to each other.

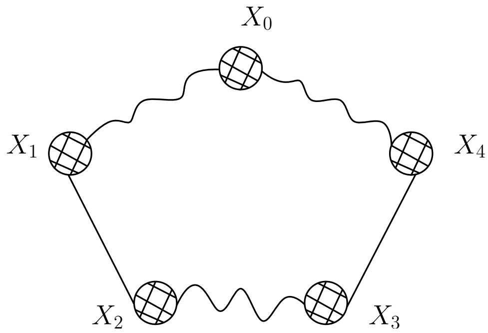

Figure 2: A 5-crown with 5-crown partition $(X_0, X_1, X_2, X_3, X_4)$ and $i^* = 0$. Crosshatched disks represent nonempty cliques. A straight line between two disks indicates that the corresponding cliques are complete to each other. A wavy line between two disks indicates that there are edges between the corresponding cliques (those edges must obey the axioms from the definition of a 5-crown). The absence of a line (straight or wavy) between two disks indicates that the corresponding cliques are anticomplete to each other.

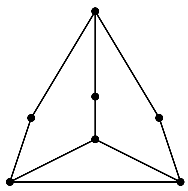

Figure 3: The 5-pyramid.

Furthermore, it is easy to see that the 5-pyramid has stability number three. Thus, the 5-pyramid is $(4K_1, C_4, C_6, C_7)$-free.

To prove Theorem 2.2, we consider two cases: the case when our graphs contain an induced 5-pyramid, and the case when they are 5-pyramid-free. More precisely, we prove the following two theorems.

Theorem 2.3. *Let G be a graph. Then the following are equivalent:*

- *G is a $(4K_1, C_4, C_6, C_7)$-free graph that contains an induced 5-pyramid and does not contain a simplicial vertex;*
- *G has exactly one nontrivial anticomponent, and this anticomponent is a 5-basket.*

Theorem 2.4. *Let G be a graph. Then the following are equivalent:*

- *G is a $(4K_1, C_4, C_6, C_7, \text{5-pyramid})$-free graph that does not contain a simplicial vertex;*

- *G has exactly one nontrivial anticomponent, and this anticomponent is a 5-crown.*

Clearly, Theorem 2.2 follows immediately from Theorems 2.3 and 2.4. We prove Theorem 2.3 in Section 2.1, and we prove Theorem 2.4 in Section 2.2. Our proof of Theorem 2.3 is from first principles; the proof of Theorem 2.4 relies heavily on certain results of [3].

2.1 Decomposing $(4K_1, C_4, C_6, C_7)$-free graphs that contain an induced 5-pyramid: proof of Theorem 2.3

We begin with a lemma that, together with Lemmas 1.6 and 2.1, establishes the "backward" implication of Theorem 2.3.

Lemma 2.5. *Every 5-basket is $(4K_1, C_4, C_6, C_7)$-free and contains an induced 5-pyramid.*

Proof. Let Q be a 5-basket, and let $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ be an associated 5-basket partition of Q.

Let us show that Q contains an induced 5-pyramid. By the definition of a 5-basket, some vertex $a_1 \in A$ is complete to $B_1 \cup B_2 \cup B_3$. We now choose an arbitrary vertex from each of the sets $B_1, B_2, B_3, C_1, C_2, C_3$, and we observe that these six vertices, together with the vertex a_1, induce a 5-pyramid in Q.

It remains to show that Q is $(4K_1, C_4, C_6, C_7)$-free. Suppose that Q is not $4K_1$-free. Then there exists a stable set of size four in Q, say $\{x, y, z, w\}$. By construction, A is complete to at least two of B_1, B_2, B_3, and F is complete to at least two of $B_1 \cup C_1, B_2 \cup C_2, B_3 \cup C_3$. So, there exists an index $i \in \{1, 2, 3\}$ such that A is complete to B_i and F is complete to $B_i \cup C_i$; by symmetry, we may assume that $i = 2$.[9] Then $(A \cup B_2 \cup F, B_1 \cup C_1, B_3 \cup C_3, C_2)$ is a partition of $V(Q)$ into four cliques; clearly, each of these four cliques contains exactly one vertex of the stable set $\{x, y, z, w\}$. By symmetry, we may assume that $x \in A \cup B_2 \cup F$, $y \in B_1 \cup C_1$, $z \in B_3 \cup C_3$, and $w \in C_2$. Since C_2 is complete to $B_2 \cup C_1 \cup C_3 \cup F$, and since $w \in C_2$ anticomplete to $\{x, y, z\}$, we in fact have that $x \in A$, $y \in B_1$, and $z \in B_3$. But this is impossible because x is anticomplete to $\{y, z\}$, and A is complete to at least one of B_1, B_3.

It remains to show that Q is (C_4, C_6, C_7)-free. Clearly, it suffices to show that all holes in Q are of length five. So, fix an integer $k \geq 4$ and a k-hole $x_0, x_1, \dots, x_{k-1}, x_0$ (with indices in $\mathbb{Z}_k$) in Q; we must show that $k = 5$.

Note first that for all $X \in \{A, B_1, B_2, B_3, C_1, C_2, C_3, F\}$, and all distinct $x, y \in X$, one of x, y dominates the other in Q. Since no vertex of a hole dominates any other vertex of that hole, we deduce that no

[9] See Figure 1.

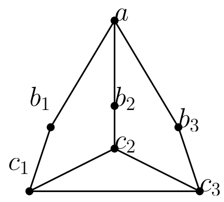

Figure 4: The 5-pyramid P with $V(P) = \{a, b_1, b_2, b_3, c_1, c_2, c_3\}$ and $E(P) = \{ab_1, ab_2, ab_3, b_1c_1, b_2c_2, b_3c_3, c_1c_2, c_2c_3, c_3c_1\}$, as in the statement of Lemma 2.6 and the proof of Lemma 2.7.

hole of Q contains more than one vertex from any one of the eight sets $A, B_1, B_2, B_3, C_1, C_2, C_3, F$.

Suppose first that F contains a vertex of our hole $x_0, x_1, \dots, x_{k-1}, x_0$; by symmetry, we may assume that $x_0 \in F$. Fix $j^* \in \{1, 2, 3\}$ such that F is complete to $V(Q) \setminus (B_{j^*} \cup C_{j^*} \cup F)$ and anticomplete to $B_{j^*} \cup C_{j^*}$. Then $x_2, \dots, x_{k-2} \in B_{j^*} \cup C_{j^*}$, and so since $B_{j^*} \cup C_{j^*}$ is a clique, we deduce that $4 \leq k \leq 5$. If $k = 5$, then we are done; so assume that $k = 4$. Then x_0, x_1, x_2, x_3, x_0 is a 4-hole in Q, with $x_0 \in F$ and $x_2 \in B_{j^*} \cup C_{j^*}$. Now, if $x_2 \in B_{j^*}$, then all the common neighbors of x_0 and x_2 are in A; and if $x_2 \in C_{j^*}$, then all the common neighbors of x_0 and x_2 are in $(C_1 \cup C_2 \cup C_3) \setminus C_{j^*}$. Thus, either $x_1, x_3 \in A$ or $x_1, x_3 \in (C_1 \cup C_2 \cup C_3) \setminus C_{j^*}$. But neither of these outcomes is possible since A and $(C_1 \cup C_2 \cup C_3) \setminus C_{j^*}$ are cliques, and x_1 and x_3 are nonadjacent.

It remains to consider the case when $x_0, x_1, \dots, x_{k-1}, x_0$ is a hole in $Q \setminus F$. It is easy to see that $Q \setminus (F \cup A)$ is chordal;[10] consequently, A contains some vertex of our hole. By symmetry, we may assume that $x_0 \in A$. It then readily follows that there exist distinct indices $i_1, i_2 \in \{1, 2, 3\}$ such that $x_1 \in B_{i_1}$ and $x_{k-1} \in B_{i_2}$. But then $x_2 \in C_{i_1}$ and $x_{k-2} \in C_{i_2}$. Since C_{i_1} and C_{i_2} are disjoint and complete to each other, we deduce that x_2 and x_{k-2} are distinct and adjacent. Thus, $k - 2 = 3$, and it follows that $k = 5$, which is what we needed to show. □

It remains to prove the "forward" implication of Theorem 2.3. We first prove a lemma that describes how vertices in a $(4K_1, C_4, C_6, C_7)$-free graph can "attach" to an induced 5-pyramid.

Lemma 2.6. *Let G be a $(4K_1, C_4, C_6, C_7)$-free graph, and let P be an induced 5-pyramid in G, with $V(P) = \{a, b_1, b_2, b_3, c_1, c_2, c_3\}$ and $E(P) =$*

[10] Let us check this. A *simplicial elimination ordering* of a graph G is an ordering $v_1, \dots, v_n$ of its vertices such that for all $i \in \{1, \dots, n\}$, v_i is simplicial in $G[v_i, v_{i+1}, \dots, v_n]$. It is well-known that a graph is chordal if and only if it admits a simplicial elimination ordering [11]. We can form a simplicial elimination ordering of $Q \setminus (A \cup F)$ by first listing all vertices of $B_1 \cup B_2 \cup B_3$ (in any order), and then listing all vertices of $C_1 \cup C_2 \cup C_3$ (again, in any order).

$\{ab_1, ab_2, ab_3, b_1c_1, b_2c_2, b_3c_3, c_1c_2, c_2c_3, c_3c_1\}$.[11] *Let* $x \in V(G) \setminus V(P)$. *Then exactly one of the following holds:*

(a) there exists an index $i \in \{1,2,3\}$ *such that* $N_G(x) \cap V(P) = \{b_i, c_i\}$*;*

(b) there exists an index $i \in \{1,2,3\}$ *such that* $N_G(x) = \{a, b_1, b_2, b_3\} \setminus \{b_i\}$*;*

(c) there exists an index $i \in \{1,2,3\}$ *such that* $N_G(x) \cap V(P) = V(P) \setminus \{b_i, c_i\}$*;*

(d) there exists a vertex $v \in V(P)$ *such that* $N_G(x) \cap V(P) = N_P[v]$*;*[12]

(e) $N_G(x) \cap V(P) = V(P)$.

Proof. It is clear that at most one of (a)-(e) can hold. It remains to show that at least one of (a)-(e) holds. We note that x is adjacent to at least one of the vertices b_1, b_2, b_3, for otherwise $\{x, b_1, b_2, b_3\}$ would be a stable set of size four in G, contrary to the fact that G is $4K_1$-free. We now consider three cases.

Case 1: x is adjacent to exactly one of b_1, b_2, b_3. By symmetry, we may assume that x is adjacent to b_1 and is nonadjacent to b_2 and b_3. Then x is adjacent to c_1, for otherwise, $\{x, c_1, b_2, b_3\}$ would be a stable set of size four in G, contrary to the fact that G is $4K_1$-free.

Suppose first that x is adjacent to a. Then x is nonadjacent to c_2, for otherwise, x, a, b_2, c_2, x would be a 4-hole in G, contrary to the fact that G is C_4-free; similarly, x is nonadjacent to c_3. We now have that $N_G(x) \cap V(P) = \{a, b_1, c_1\}$, and so x satisfies (d) for $v = b_1$.

Suppose now that x is nonadjacent to a. If x is adjacent neither to c_2 nor to c_3, then x satisfies (a) for $i = 1$, and we are done. On the other hand, if x is adjacent both to c_2 and to c_3, then x satisfies (d) for $v = c_1$, and again we are done. It remains to consider the case when x is adjacent to exactly one of c_2, c_3; by symmetry, we may assume that x is adjacent to c_2 and nonadjacent to c_3. But now $a, b_1, x, c_2, c_3, b_3, a$ is a 6-hole in G, contrary to the fact that G is C_6-free.

Case 2: x is adjacent to exactly two of b_1, b_2, b_3. By symmetry, we may assume that x is adjacent to b_1, b_2 and nonadjacent to b_3. Then x is adjacent to a, for otherwise, x, b_1, a, b_2, x would be a 4-hole in G, contrary to the fact that G is C_4-free. Furthermore, x is nonadjacent to c_3, for otherwise, x, a, b_3, c_3, x would be a 4-hole in G, again contrary to the fact that G is C_4-free. If x adjacent neither to c_1 nor to c_2, then x satisfies (b) for $i = 3$, and we are done. On the other hand, if x is adjacent both to c_1 and to c_2, then x satisfies (c) for $i = 3$, and again we are done. It remains to consider the case when x is adjacent to exactly one of c_1, c_2; by symmetry,

[11] See Figure 4.

[12] Note that this means that x is adjacent to v and to all neighbors of v in P (and to no other vertices of P).

we may assume that x is adjacent to c_1 and nonadjacent to c_2. But now x, c_1, c_2, b_2, x is a 4-hole in G, contrary to the fact that G is C_4-free.

Case 3: x adjacent to all three of b_1, b_2, b_3. Then x is adjacent to a, for otherwise, x, b_1, a, b_2, x would be a 4-hole in G, contrary to the fact that G is C_4-free. If x is adjacent to none of c_1, c_2, c_3, then x satisfies (d) for $v = a$, and we are done. On the other hand, if x is adjacent to all of c_1, c_2, c_3, then x satisfies (e), and again we are done. It remains to consider the case when x has at least one neighbor and at least one nonneighbor in $\{c_1, c_2, c_3\}$; by symmetry, we may assume that x is adjacent to c_1 and nonadjacent to c_2. But now x, c_1, c_2, b_2, x is a 4-hole in G, contrary to the fact that G is C_4-free. □

We now prove the "forward" implication of Theorem 2.3.

Lemma 2.7. *Every $(4K_1, C_4, C_6, C_7)$-free graph G that contains an induced 5-pyramid satisfies exactly one of the following:*

- *G has exactly one nontrivial anticomponent, and this anticomponent is a 5-basket;*
- *G contains a simplicial vertex.*

Proof. By Lemma 2.1, 5-baskets do not contain simplicial vertices, and so by Lemma 1.6(a), no graph satisfies both outcomes of the statement of Lemma 2.7. It remains to show that $(4K_1, C_4, C_6, C_7)$-free graphs that contain an induced 5-pyramid satisfy at least one of those two outcomes.

Let G be a $(4K_1, C_4, C_6, C_7)$-free graph, and let P be an induced 5-pyramid in G. Set $V(P) = \{a, b_1, b_2, b_3, c_1, c_2, c_3\}$ and

$$E(P) = \{ab_1, ab_2, ab_3, b_1c_1, b_2c_2, b_3c_3, c_1c_2, c_2c_3, c_3c_1\},$$

as in Figure 4. We now construct several sets, as follows. First, for all $i \in \{1, 2, 3\}$, we define sets H_i, T_i, F_i as follows:

- $H_i = \Big\{x \in V(G) \setminus V(P) \mid N_G(x) \cap V(P) = \{b_i, c_i\}\Big\}$;
- $T_i = \Big\{x \in V(G) \setminus V(P) \mid N_G(x) \cap V(P) = \{a, b_1, b_2, b_3\} \setminus \{b_i\}\Big\}$;
- $F_i = \Big\{x \in V(G) \setminus V(P) \mid N_G(x) \cap V(P) = V(P) \setminus \{b_i, c_i\}\Big\}$.

Further, for all $v \in V(P)$, we set

- $C_v = \{x \in V(G) \mid N_G[x] \cap V(P) = N_P[v]\}$.[13]

Finally, we set

[13]Thus, $v \in C_v$, and furthermore, $C_v \setminus \{v\}$ is precisely the set of all vertices in $V(G) \setminus V(P)$ that satisfy (d) from Lemma 2.6 for our choice of v.

- $Z = \{x \in V(G) \mid N_G(x) \cap V(P) = V(P)\}$.

  Claim 1. *The sets*

  $H_1, H_2, H_3, T_1, T_2, T_3, F_1, F_2, F_3, C_a, C_{b_1}, C_{b_2}, C_{b_3}, C_{c_1}, C_{c_2}, C_{c_3}, Z$

  are pairwise disjoint, and their union is $V(G)$*.*

Proof of Claim 1. This follows from the construction and from Lemma 2.6. ■

We now define the following sets:

- let $A = C_a \cup T_1 \cup T_2 \cup T_3$;
- for all $i \in \{1, 2, 3\}$, let H'_i be the set of all vertices in H_i that are anticomplete to A;
- for all $i \in \{1, 2, 3\}$, let $B_i = C_{b_i} \cup (H_i \setminus H'_i)$;[14]
- for all $i \in \{1, 2, 3\}$, let $C_i = C_{c_i}$;
- let $F = F_1 \cup F_2 \cup F_3$;
- let $H = H'_1 \cup H'_2 \cup H'_3$.

Further, recall that Z is the set of all vertices in $V(G) \setminus V(P)$ that are complete to $V(P)$ in G. Set $Q = G \setminus (H \cup Z)$. Our goal is to prove the following:[15]

- if $H = \emptyset$, then Q is the only nontrivial anticomponent of G, and Q is a 5-basket with 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$;
- if $H \neq \emptyset$, and if $h \in H$ is chosen so that $d_G(h)$ is as small as possible, then h is a simplicial vertex of G.

We do this by proving a sequence of claims.

> ***Claim 2.*** *Sets* $A, B_1, B_2, B_3, C_1, C_2, C_3, F, Z, H$ *are pairwise disjoint, and their union is precisely* $V(G)$*. Furthermore,* $a \in A$*, and for all* $i \in \{1, 2, 3\}$*, we have* $b_i \in B_i$ *and* $c_i \in C_i$*. In particular, sets* $A, B_1, B_2, B_3, C_1, C_2, C_3$ *are all nonempty.*

Proof of Claim 2. This follows from Claim 1 and from the construction. ■

[14] Note that this implies that all vertices in B_i have a neighbor in A. Indeed, all vertices in C_{b_i} are adjacent to $a \in C_a \subseteq A$, and by construction, all vertices in $H_i \setminus H'_i$ have a neighbor in A.

[15] See Claims 11, 12, and 13.

Claim 3. *At most one of the sets $H_1 \cup T_1$, $H_2 \cup T_2$, and $H_3 \cup T_3$ is nonempty. There exists some $i \in \{1,2,3\}$ such that $H = H_i'$. For all $i \in \{1,2,3\}$, H_i' is complete to $(B_i \cup C_i)$ and anticomplete to $\Big(V(Q) \setminus (B_i \cup C_i \cup F)\Big) \cup F_i$.*[16] *At most one of the sets F_1, F_2, and F_3 is nonempty, and consequently, there exists some $j \in \{1,2,3\}$ such that $F = F_j$.*

Proof of Claim 3. Suppose that at least two of the sets H_1, H_2, H_3 are nonempty. By symmetry, we may assume that H_1, H_2 are both nonempty; fix $h_1 \in H_1$ and $h_2 \in H_2$. But now if h_1, h_2 are adjacent, then h_1, c_1, c_2, h_2, h_1 is a 4-hole in G, contrary to the fact that G is C_4-free; and if h_1, h_2 are nonadjacent, then $\{a, h_1, h_2, c_3\}$ is a stable set of size four in G, contrary to the fact that G is $4K_1$-free. This proves that at most one of the sets H_1, H_2, H_3 is nonempty.

Suppose that at least two of the sets T_1, T_2, T_3 are nonempty. By symmetry, we may assume that T_1, T_2 are both nonempty; fix $t_1 \in T_1$ and $t_2 \in T_2$. But now if t_1, t_2 are adjacent, then $t_1, b_2, c_2, c_1, b_1, t_2, t_1$ is a 6-hole in G, contrary to the fact that G is C_6-free; and if t_1, t_2 are nonadjacent, then $t_1, b_2, c_2, c_1, b_1, t_2, b_3, t_1$ is a 7-hole in G, contrary to the fact that G is C_7-free. This proves that at most one of the sets T_1, T_2, T_3 is nonempty.

We now show that at most one of the sets $H_1 \cup T_1$, $H_2 \cup T_2$, and $H_3 \cup T_3$ is nonempty. Suppose otherwise. By what we just showed, and by symmetry, we may assume that H_1 and T_3 are both nonempty. Fix $h_1 \in H_1$ and $t_3 \in T_3$. But now if h_1, t_3 are adjacent, then $a, t_3, h_1, c_1, c_3, b_3, a$ is a 6-hole in G, contrary to the fact that G is C_6-free; and if h_1, t_3 are nonadjacent, then $\{h_1, t_3, c_2, b_3\}$ is a stable set of size four in G, contrary to the fact that G is $4K_1$-free. This proves the first statement of Claim 3.

The second statement of Claim 3 follows from the first statement, and from the construction.

We now prove the third statement of Claim 3. By symmetry, it suffices to show that H_1' is complete to $B_1 \cup C_1$ and anticomplete to $A \cup B_2 \cup B_3 \cup C_2 \cup C_3 \cup F_1$. If some $h_1 \in H_1'$ and $x \in B_1 \cup C_1$ are nonadjacent, then $\{h_1, x, b_2, b_3\}$ is a stable set of size four in G, contrary to the fact that G is $4K_1$-free. Thus, H_1' is complete to $B_1 \cup C_1$. It remains to show that H_1' is anticomplete to $A \cup B_2 \cup B_3 \cup C_2 \cup C_3 \cup F_1$. First, by construction, H_1' is anticomplete to A. Next, if some $h_1 \in H_1'$ and $b_2' \in B_2$ are adjacent, then h_1, c_1, c_2, b_2', h_1 is a 4-hole in G, contrary to the fact that G is C_4-free. So, H_1' is anticomplete to B_2, and similarly, H_1' is anticomplete to B_3. If some $h_1 \in H_1'$ and $c_2' \in C_2$ are adjacent, then $h_1, c_2', c_3, b_3, a, b_1, h_1$ is a 6-hole in G, contrary to the fact that G is C_6-free. Thus, H_1' is anticomplete to C_2, and similarly, H_1' is anticomplete to C_3. If some $h_1 \in H_1'$ and $f_1 \in F_1$ are adjacent, then

[16] Note that $\Big(V(Q) \setminus (B_i \cup C_i \cup F)\Big) \cup F_i = \Big((A \cup B_1 \cup B_2 \cup B_3 \cup C_1 \cup C_2 \cup C_3) \setminus (B_i \cup C_i)\Big) \cup F_i$.

h_1, c_1, c_2, f_1, h_1 is a 4-hole in G, contrary to the fact that G is C_4-free. We have now shown that H'_1 is anticomplete to $A \cup B_2 \cup B_3 \cup C_2 \cup C_3 \cup F_1$. This proves the third statement of Claim 3.

Suppose that at least two of F_1, F_2, F_3 are nonempty. By symmetry, we may assume that F_1, F_2 are both nonempty; fix $f_1 \in F_1$ and $f_2 \in F_2$. But then if f_1, f_2 are adjacent, then f_1, c_2, c_1, f_2, f_1 is a 4-hole in G, contrary to the fact that G is C_4-free; and if f_1, f_2 are nonadjacent, then a, f_1, c_3, f_2, a is a 4-hole in G, again contrary to the fact that G is C_4-free. This proves that at most one of F_1, F_2, F_3 is nonempty. By construction, we have that $F = F_1 \cup F_2 \cup F_3$, and so it follows that there exists some $j \in \{1, 2, 3\}$ such that $F = F_j$. This completes the proof of Claim 3. ■

> ***Claim 4.*** *Sets $A, B_1, B_2, B_3, C_1, C_2, C_3, F, Z, H$ are all cliques.*

Proof of Claim 4. We first prove that A is a clique. By Claim 3, at most one of the sets T_1, T_2, T_3 is nonempty. So, by construction and by symmetry, we may assume that $A = C_a \cup T_3$, so that A is complete to $\{b_1, b_2\}$. But now if some $a_1, a_2 \in A$ are nonadjacent, then a_1, b_1, a_2, b_2, a_1 is a 4-hole in G, contrary to the fact that G is C_4-free. This proves that A is a clique.

If some $x, y \in B_1$ are nonadjacent, then $\{x, y, b_2, b_3\}$ is a stable set of size four in G, contrary to the fact that G is $4K_1$-free. Thus, B_1 is a clique. Similarly, B_2 and B_3 are cliques.

If some $x, y \in C_1$ are nonadjacent, then b_1, x, c_2, y, b_1 is a 4-hole in G, contrary to the fact that G is C_4-free. Thus, C_1 is a clique. Similarly, C_2 and C_3 are cliques.

If some $x, y \in F_3$ are nonadjacent, then a, x, c_1, y, a is a 4-hole in G, contrary to the fact that G is C_4-free. Thus, F_3 is a clique. Similarly, F_1 and F_2 are cliques. Since $F = F_1 \cup F_2 \cup F_3$, and since (by Claim 3) at most one of F_1, F_2, F_3 is nonempty, we deduce that F is a clique.

If some $z_1, z_2 \in Z$ are nonadjacent, then a, z_1, c_1, z_2, a is a 4-hole in G, contrary to the fact that G is C_4-free. Thus, Z is a clique.

If some $x, y \in H_1$ are nonadjacent, then $\{x, y, b_2, b_3\}$ is a stable set of size four in G, contrary to the fact that G is $4K_1$-free. Thus, H_1 is a clique, and similarly, H_2 and H_3 are cliques. It follows that H'_1, H'_2, H'_3 are cliques, and so Claim 3 implies that H is a clique. This proves Claim 4. ■

> ***Claim 5.*** *A is anticomplete to C_1, C_2, C_3. Sets B_1, B_2, B_3 are anticomplete to each other. Sets C_1, C_2, C_3 are complete to each other. For all $i \in \{1, 2, 3\}$, B_i is complete to C_i and anticomplete to $(C_1 \cup C_2 \cup C_3) \setminus C_i$.*

Proof of Claim 5. Suppose that some $a' \in A$ and $c'_1 \in C_1$ are adjacent. By the construction of A, we see that a' is adjacent to at least one of b_2, b_3, and that a' is anticomplete to $\{c_1, c_2, c_3\}$. By symmetry, we may assume that a'

is adjacent to b_2. But now a', b_2, c_2, c_1', a' is a 4-hole in G, contrary to the fact that G is C_4-free. Thus, A is anticomplete to C_1, and similarly, A is anticomplete to C_2, C_3.

If some $b_1' \in B_1$ and $b_2' \in B_2$ are adjacent, then $b_1', c_1, c_2, b_2', b_1'$ is a 4-hole in G, contrary to the fact that G is C_4-free. Thus, B_1 is anticomplete to B_2. By symmetry, it follows that B_1, B_2, B_3 are anticomplete to each other.

If some $c_1' \in C_1$ and $c_2' \in C_2$ are nonadjacent, then $a, b_1, c_1', c_3, c_2', b_2, a$ is a 6-hole in G, contrary to the fact that G is C_6-free. Thus, C_1 is complete to C_2. By symmetry, it follows that C_1, C_2, C_3 are complete to each other.

It remains to prove the last statement of Claim 5. By symmetry, it suffices to show that B_1 is complete to C_1 and anticomplete to $C_2 \cup C_3$.

If some $b_1' \in B_1$ and $c_1' \in C_1$ are nonadjacent, then $\{b_1', c_1', b_2, b_3\}$ is a stable set of size four in G, contrary to the fact that G is $4K_1$-free. Thus, B_1 is complete to C_1.

Next, suppose some $b_1' \in B_1$ is adjacent to some $c_2' \in C_2$. By the construction of B_1, b_1' is adjacent to some $a' \in A$. We have already shown that A is anticomplete to C_2, and so a' and c_2' are nonadjacent. Further, by construction, b_1' and b_2 are nonadjacent. So, if a' is adjacent to b_2, then a', b_1', c_2', b_2, a' is a 4-hole in G, contrary to the fact that G is C_4-free. This proves that a' is nonadjacent to b_2. It then follows from the construction of A that $a' \in T_2$. So, by Claim 3, H_1 is empty; consequently, $B_1 = C_{b_1}$, and we have that $b_1' \in C_{b_1}$. But now a, b_1', c_2', b_2, a is a 4-hole in G, contrary to the fact that G is C_4-free. Thus, B_1 is anticomplete to C_2; similarly, B_1 is anticomplete to C_3. This proves Claim 5. ■

Claim 6. *For all $j \in \{1, 2, 3\}$, $Z \cup F_j$ is complete to $(A \cup B_1 \cup B_2 \cup B_3 \cup C_1 \cup C_2 \cup C_3) \setminus (B_j \cup C_j)$.*

Proof of Claim 6. By symmetry, it suffices to prove the statement for $j = 3$, *i.e.* to show that $Z \cup F_3$ is complete to $A \cup B_1 \cup B_2 \cup C_1 \cup C_2$. Fix $x \in Z \cup F_3$; then x is complete to $\{a, b_1, b_2, c_1, c_2\}$, and x is either complete or anticomplete to $\{b_3, c_3\}$. We must show that x is complete to $A \cup B_1 \cup B_2 \cup C_1 \cup C_2$. Recall that $A = C_a \cup T_1 \cup T_2 \cup T_3$.

First, note that every vertex in $T_3 \cup C_1 \cup C_2$ has two nonadjacent neighbors in $\{a, b_1, b_2, c_1, c_2\}$. So, if x is nonadjacent to some $y \in T_3 \cup C_1 \cup C_2$, then we fix distinct, nonadjacent neighbors $u, v \in \{a, b_1, b_2, c_1, c_2\}$ of y, and we observe that x, u, y, v, x is a 4-hole in G, contrary to the fact that G is C_4-free. So, x is complete to $T_3 \cup C_1 \cup C_2$.

Next, suppose that x is nonadjacent to some $t_1 \in T_1$. If x is anticomplete to $\{b_3, c_3\}$, then $x, c_1, c_3, b_3, t_1, b_2, x$ is a 6-hole in G, contrary to the fact that G is C_6-free. So, x is complete to $\{b_3, c_3\}$. But now x, b_2, t_1, b_3, x is a 4-hole in G, contrary to the fact that G is C_4-free. Thus, x is complete to T_1, and similarly, x is complete to T_2.

So far, we have shown that x is complete to $A \cup C_1 \cup C_2$. It remains to show that x is complete to $B_1 \cup B_2$. Suppose otherwise. By symmetry, we may assume that x is nonadjacent to some $b'_1 \in B_1$. By the construction of B_1, b'_1 is adjacent to some $a' \in A$. We have already shown that x is complete to A; so, x is adjacent to a'. But now a', b'_1, c_1, x, a' is a 4-hole in G, contrary to the fact that G is C_4-free. This proves Claim 6. ■

Claim 7. *There exists some $j \in \{1,2,3\}$ such that F is complete to $(A \cup B_1 \cup B_2 \cup B_3 \cup C_1 \cup C_2 \cup C_3) \setminus (B_j \cup C_j)$ and anticomplete to $B_j \cup C_j$.*

Proof of Claim 7. By Claim 3, there exists some $j \in \{1,2,3\}$ such that $F = F_j$; by symmetry, we may assume that $j = 3$. Then by Claim 6, F is complete to $A \cup B_1 \cup B_2 \cup C_1 \cup C_2$. It remains to show that F is anticomplete to $B_3 \cup C_3$. If some $f_3 \in F$ and $c'_3 \in C_3$ are adjacent, then a, f_3, c'_3, b_3, a is a 4-hole in G, contrary to the fact that G is C_4-free. On the other hand, if f_3 is adjacent to some $b'_3 \in B_3$, then f_3, b'_3, c_3, c_2, f_3 is a 4-hole in G, again contrary to the fact that G is C_4-free. So, F is anticomplete to $B_3 \cup C_3$. This proves Claim 7. ■

Claim 8. *Z is complete to $V(G) \setminus (Z \cup H)$.*

Proof of Claim 8. By Claim 6, Z is complete to $A \cup B_1 \cup B_2 \cup B_3 \cup C_1 \cup C_2 \cup C_3$. In view of Claim 2, it remains to show that Z is complete to F. By Claim 3, there exists some $j \in \{1,2,3\}$ such that $F = F_j$; by symmetry, we may assume that $j = 3$, so that $F = F_3$. Now, suppose that some $z \in Z$ and $f_3 \in F_3$ are nonadjacent. Then z, b_1, f_3, b_2, z is a 4-hole in G, contrary to the fact that G is C_4-free. This proves Claim 8. ■

Claim 9. *A is complete to at least two of the sets B_1, B_2, B_3.*

Proof of Claim 9. Suppose otherwise. By symmetry, we may assume that A is complete neither to B_1 nor to B_2. Fix $a_1, a_2 \in A$,[17] $b'_1 \in B_1$, and $b'_2 \in B_2$ such that a_1 is nonadjacent to b'_1, and a_2 is nonadjacent to b'_2. By Claim 5, b'_1 and b'_2 are nonadjacent. If a_1 is nonadjacent to b'_2, then $\{a_1, b'_1, b'_2, c_3\}$ is a stable set of size four in G, contrary to the fact that G is $4K_1$-free. Thus, a_1 is adjacent to b'_2, and similarly, a_2 is adjacent to b'_1; in particular, $a_1 \neq a_2$. Since A is a clique (by Claim 4), we see that a_1, a_2 are adjacent. But now $a_1, a_2, b'_1, c_1, c_2, b'_2, a_1$ is a 6-hole in G, contrary to the fact that G is C_6-free. This proves Claim 9. ■

Claim 10. *For all $i \in \{1,2,3\}$, A can be ordered as $A = \{a_1, \dots, a_t\}$ so that $N_G(a_t) \cap B_i \subseteq \dots \subseteq N_G(a_1) \cap B_i = B_i$.*

[17]Vertices a_1, a_2 need not be distinct.

Proof of Claim 10. Fix $i \in \{1, 2, 3\}$. Suppose that for some $x, y \in A$, neither one of $N_G(x) \cap B_i$ and $N_G(y) \cap B_i$ is included in the other. Fix $b_x, b_y \in B_i$ such that x is adjacent to b_x and nonadjacent to b_y, and y is adjacent to b_y and nonadjacent to b_x. (In particular, $x \neq y$ and $b_x \neq b_y$.) By Claim 4, A and B_i are both cliques, and we deduce that x, b_x, b_y, y, x is a 4-hole in G, contrary to the fact that G is C_4-free. This proves that for all $i \in \{1, 2, 3\}$, A can be ordered as $A = \{a_1, \dots, a_t\}$ so that $N_G(a_t) \cap B_i \subseteq \dots \subseteq N_G(a_1) \cap B_i$.

It remains to show that $N_G(a_1) \cap B_i = B_i$, *i.e.* that a_1 is complete to B_i. Fix $b'_i \in B_i$. By the construction of B_i, b'_i is adjacent to some $a' \in A$. But now $b'_i \in N_G(a') \cap B_i \subseteq N_G(a_1) \cap B_i$, and so a_1 is adjacent to b'_i. This proves Claim 10. ■

> ***Claim 11.*** *Q is a 5-basket, and $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ is a 5-basket partition for it.*

Proof of Claim 11. By construction, $Q = G \setminus (H \cup Z)$. The fact that Q is a 5-basket with 5-basket partition $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ now follows from Claims 2, 4, 5, 7, 9, and 10. This proves Claim 11. ■

> ***Claim 12.*** *If $H = \emptyset$, then Q is the only nontrivial anticomponent of G.*

Proof of Claim 12. Assume that $H = \emptyset$. Then by Claim 2, we have that $V(G) \setminus V(Q) = Z$. By Claim 4, Z is a clique, and by Claim 8, Z complete to $V(Q)$. By Claim 11, Q is a 5-basket, and by Lemma 2.1, all 5-baskets are anticonnected. So, Q is the only nontrivial anticomponent of G. This proves Claim 12. ■

> ***Claim 13.*** *If $H \neq \emptyset$, and if $h \in H$ is chosen so that $d_G(h)$ is as small as possible, then h is a simplicial vertex of G.*

Proof of Claim 13. Assume that $H \neq \emptyset$, and let $h \in H$ be chosen so that $d_G(h)$ is as small as possible. We must show that h is simplicial in G. By Claim 3, and by symmetry, we may assume that $H = H'_1$. Further by Claim 3, we have that H is complete to $B_1 \cup C_1$ and anticomplete to $\Big(V(Q) \setminus (B_1 \cup C_1 \cup F)\Big) \cup F_1$. So,

$$\Big(H, \Big(V(Q) \setminus (B_1 \cup C_1 \cup F)\Big) \cup F_1, B_1 \cup C_1 \cup (F \setminus F_1) \cup Z\Big)$$

is a cut-partition of G.[18] Furthermore, Claims 4, 5, 6, and 8 together guarantee that $B_1 \cup C_1 \cup (F \setminus F_1) \cup Z$ is a clique, and so our cut-partition of G is in fact a clique-cut-partition of G. Since H is a clique (by Claim 4), Lemma 1.7 now implies that h is simplicial in G. This proves Claim 13. ■

In view of Claims 11, 12, and 13, the proof is complete. □

[18] Note that we are implicitly using Claim 2.

We are now ready to prove Theorem 2.3, restated below for the reader's convenience.

Theorem 2.3. *Let G be a graph. Then the following are equivalent:*

- *G is a $(4K_1, C_4, C_6, C_7)$-free graph that contains an induced 5-pyramid and does not contain a simplicial vertex;*
- *G has exactly one nontrivial anticomponent, and this anticomponent is a 5-basket.*

Proof. The "forward" implication follows from Lemma 2.7. The "backward" implication follows from Lemmas 1.6, 2.1, and 2.5. □

2.2 Decomposing $(4K_1, C_4, C_6, C_7, 5$-pyramid$)$-free graphs: proof of Theorem 2.4

In this section, we use the results of [3] to prove Theorem 2.4. We remind the reader that "rings" and "5-crowns" were defined at the beginning of Section 2. Our first goal is to prove that $(4K_1, C_4, C_6, C_7)$-free rings are precisely the 5-crowns (see Lemma 2.11).

The following is Lemma 2.4(b) from [3].

Lemma 2.8. *[3] Let $k \geq 4$ be an integer. Then every hole in a k-ring is of length k.*

Lemma 2.9. *Every ring contains a hole of the same length as the ring itself, and no ring contains a simplicial vertex.*

Proof. Let R be a k-ring ($k \geq 4$) with ring partition $(X_0, \dots, X_{k-1})$. For all $i \in \mathbb{Z}_k$, set $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ so that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$, as in the definition of a ring. Then $u_0^1, u_1^1, \dots, u_{k-1}^1, u_0^1$ is a k-hole in R. Furthermore, for all $i \in \mathbb{Z}_k$, every vertex in X_i is complete to $\{u_{i-1}^1, u_{i+1}^1\}$, and u_{i-1}^1, u_{i+1}^1 are nonadjacent. So, R does not contain any simplicial vertices. □

Lemma 2.10. *Let R be a graph. Then the following are equivalent:*

- *R is a $4K_1$-free 5-ring;*
- *R is a 5-crown.*

Moreover, any ring partition of a $4K_1$-free 5-ring is a 5-crown partition.

Proof. By definition, every 5-crown is a 5-ring. Furthermore, the vertex set of any 5-crown can be partitioned into three cliques; consequently all 5-crowns are $4K_1$-free. So, if R is a 5-crown, then R is a $4K_1$-free 5-ring.

Conversely, suppose that R is a $4K_1$-free 5-ring with ring partition $(X_0, \dots, X_4)$. We must show that R is a 5-crown with 5-crown partition

$(X_0, \dots, X_4)$. For all $i \in \mathbb{Z}_5$, let $X_i = \{u_i^1, \dots, u_i^{|X_i|}\}$ be an ordering of X_i such that $X_i \subseteq N_R[u_i^{|X_i|}] \subseteq \dots \subseteq N_R[u_i^1] = X_{i-1} \cup X_i \cup X_{i+1}$, as in the definition of a 5-ring. We may assume that there exists an index $i \in \mathbb{Z}_5$ such that X_i is complete neither to X_{i-1} nor to X_{i+1}, for otherwise, the result is immediate. By symmetry, we may assume that X_0 is complete neither to X_4 nor to X_1. It now follows from the orderings of the sets X_4, X_0, X_1 that $\{u_4^{|X_4|}, u_0^{|X_0|}, u_1^{|X_1|}\}$ is a stable set of size three in R. Since R is $4K_1$-free, and since $u_3^{|X_3|}$ is nonadjacent to $u_0^{|X_0|}, u_1^{|X_1|}$, we deduce that $u_3^{|X_3|}$ is adjacent to $u_4^{|X_4|}$; it then follows from the orderings of X_3, X_4 that X_3 is complete to X_4. Similarly, X_2 is complete to X_1. Thus, R is a 5-crown with 5-crown partition $(X_0, \dots, X_4)$. □

Lemma 2.11. *Let R be a graph. Then the following are equivalent:*

- *R is a $(4K_1, C_4, C_6, C_7)$-free ring;*
- *R is a 5-crown.*

Proof. The "backward" implication follows from Lemmas 2.8 and 2.10. To prove the "forward" implication, we suppose that R is a $(4K_1, C_4, C_6, C_7)$-free ring. Let k be the length of the ring R (so, $k \geq 4$). By Lemma 2.9, R contains a k-hole. On the other hand, since R is $(4K_1, C_4, C_6, C_7)$-free, we see that all holes in R are of length five. Thus, $k = 5$, that is, R is a 5-ring. Since R is $4K_1$-free, Lemma 2.10 now implies that R is a 5-crown. □

We now need a few definitions. A *theta* is any subdivision of the complete bipartite graph $K_{2,3}$; in particular, $K_{2,3}$ is a theta. A *pyramid* is any subdivision of the complete graph K_4 in which one triangle remains unsubdivided, and of the remaining three edges, at least two edges are subdivided at least once.[19] A *prism* is any subdivision of $\overline{C_6}$ (where $\overline{C_6}$ is the complement of C_6) in which the two triangles remain unsubdivided; in particular, $\overline{C_6}$ is a prism. A *three-path-configuration* (or *3PC* for short) is any theta, pyramid, or prism; the three types of 3PC are represented in Figure 5.

A *wheel* is a graph that consists of a hole and an additional vertex that has at least three neighbors in the hole (see Figure 6). If this additional vertex is adjacent to all vertices of the hole, then the wheel is said to be a *universal wheel*; if the additional vertex is adjacent to three consecutive vertices of the hole, and to no other vertices of the hole, then the wheel is said to be a *twin wheel*. A *proper wheel* is a wheel that is neither a universal wheel nor a twin wheel.

A *Truemper configuration* is any 3PC or wheel. Classes defined by forbidding various combinations of Truemper configurations as induced subgraphs have received a great deal of attention in recent years (see [16] for a slightly

[19] Note that the 5-pyramid (see Figure 3) is a special type of pyramid.

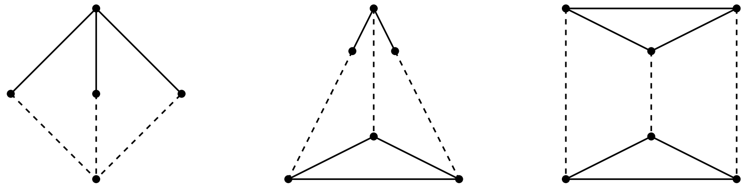

Figure 5: Three-path-configurations: theta (left), pyramid (center), and prism (right). A full line represents an edge, and a dashed line represents a path that has at least one edge.

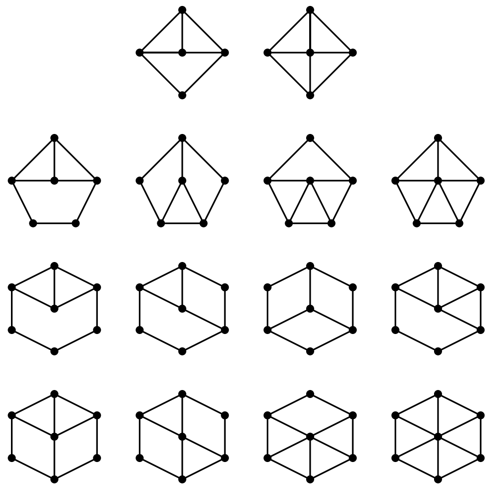

Figure 6: Some small wheels.

dated survey). Here, we are interested in the class of (3PC, proper wheel)-free graphs; this class is called $\mathcal{G}_{\text{UT}}$, and it was originally introduced in [3]. The following is Lemma 2.4(d) from [3].

Lemma 2.12. *[3] Every ring is (3PC, proper wheel, universal wheel)-free.*

Lemma 2.13. *Every* $(4K_1, C_4, C_6, C_7, 5\text{-}pyramid)$*-free graph belongs to* $\mathcal{G}_{UT}$.

Proof. Let G be a $(4K_1, C_4, C_6, C_7, 5\text{-pyramid})$-free graph. Then every hole in G is of length five; in particular, G is even-hole-free. Note that every theta and every prism contains an even hole; consequently, G is (theta, prism)-free. On the other hand, it is easy to see that the 5-pyramid is the only pyramid in which all holes are of length five. Since G is 5-pyramid-free, it follows that G is pyramid-free. Thus, G is 3PC-free. Finally, we observe that every proper wheel contains two holes of different length; since all holes in G are of length five, we deduce that G is proper-wheel-free. It now follows that $G \in \mathcal{G}_{\text{UT}}$. $\square$

A *hyperhole* is a graph H whose vertex set can be partitioned into $k \geq 4$ nonempty cliques, say $X_0, \dots, X_{k-1}$ (with indices understood to be in $\mathbb{Z}_k$), such that for all $i \in \mathbb{Z}_k$, X_i is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$. Under these circumstances, we also say that H is a hyperhole of *length* k, as well as that H is a k*-hyperhole*. Note that all k-hyperholes are k-rings.

The following decomposition theorem for $\mathcal{G}_{\text{UT}}$ was proven in [3]. (A *long hole* is a hole of length at least five.)

Theorem 2.14. *[3] Let* $G \in \mathcal{G}_{UT}$. *Then one of the following holds:*

- *G has exactly one nontrivial anticomponent, and this anticomponent is a long ring;*[20]
- *G is (long hole, $K_{2,3}$, $\overline{C_6}$)-free;*
- *$\alpha(G) = 2$, and every anticomponent of G is either a 5-hyperhole or a $(C_5, \overline{C_6})$-free graph;*
- *G admits a clique-cutset.*

We are now ready to prove Theorem 2.4, restated below for the reader's convenience.

Theorem 2.4. *Let G be a graph. Then the following are equivalent:*

- *G is a $(4K_1, C_4, C_6, C_7, 5\text{-}pyramid)$-free graph that does not contain a simplicial vertex;*

[20] We remind the reader that a ring is *long* if it is of length at least five.

- *G has exactly one nontrivial anticomponent, and this anticomponent is a 5-crown.*

Proof. We first prove the "backward" implication. So, suppose that G has exactly one nontrivial anticomponent, call it Q, and assume that Q is a 5-crown. The fact that Q is $(4K_1, C_4, C_6, C_7)$-free follows from Lemma 2.11. Further, since every 5-crown is a 5-ring, Lemma 2.12 guarantees that Q is 3PC-free; in particular, Q is 5-pyramid-free. We have now shown that Q is $(4K_1, C_4, C_6, C_7, \text{5-pyramid})$-free. Further, by Lemma 2.1, Q has no simplicial vertices. The fact that G is $(4K_1, C_4, C_6, C_7, \text{5-pyramid})$-free and contains no simplicial vertices now follows from Lemma 1.6(a).

It remains to prove the "forward" implication. So, suppose that G is $(4K_1, C_4, C_6, C_7, \text{5-pyramid})$-free and does not contain a simplicial vertex. By Lemma 2.13, we have that $G \in \mathcal{G}_{\text{UT}}$, and by Lemma 1.7, G does not admit a clique-cutset. Theorem 2.14 now implies that G satisfies at least one of the following:

(a) G has exactly one nontrivial anticomponent, and this anticomponent is a long ring;

(b) G is (long hole, $K_{2,3}$, $\overline{C_6}$)-free;

(c) $\alpha(G) = 2$, and every anticomponent of G is either a 5-hyperhole or a $(C_5, \overline{C_6})$-free graph.

If G satisfies (a), then Lemma 2.11 implies that the only nontrivial anticomponent of G is a 5-crown.

Suppose next that G satisfies (b). Then G is both long-hole-free and C_4-free; consequently, G contains no holes, *i.e.* G is chordal. But then G contains a simplicial vertex [11], a contradiction.

Suppose finally that G satisfies (c). By Lemma 1.5, G contains exactly one nontrivial anticomponent, call it Q. Since G satisfies (c), we have that Q is either a 5-hyperhole or a $(C_5, \overline{C_6})$-free graph. If Q is a 5-hyperhole, then Q is a 5-crown, and we are done. So assume that Q is a $(C_5, \overline{C_6})$-free graph. Since Q is $(4K_1, C_4, C_6, C_7)$-free, we know that all holes in Q are of length five; since Q is C_5-free, we deduce that Q contains no holes, *i.e.* that Q is chordal. Consequently (by [11]), Q contains a simplicial vertex. But now by Lemma 1.6(a), G contains a simplicial vertex, a contradiction. $\square$

3 On the clique-width of $(4K_1, C_4, C_6, C_7)$-free graphs: proof of Theorem 1.3

A *labeling* of a graph G is any function whose domain is $V(G)$. A *labeled graph* is an ordered pair (G, L), where G is a graph, and L is a labeling of G; for a vertex $v \in V(G)$, $L(v)$ is the *label* of v. The *disjoint union* of two labeled graphs on disjoint vertex sets is defined in the natural way. To simplify notation, for a labeled graph (G, L) and an induced subgraph H of G, we often write (H, L) instead of $(H, L \upharpoonright V(H))$.[21]

The *clique-width* of a labeled graph (G, L), denoted by $\text{cwd}(G, L)$, is the minimum number of labels needed to construct (G, L) using the following four operations:[22]

1. creation of a new vertex v with label i;
2. disjoint union of two labeled graphs;
3. joining by an edge every vertex labeled i to every vertex labeled j (where $i \neq j$);
4. renaming label i to label j.

Thus, at the end of the procedure, each vertex $v \in V(G)$ is supposed to have label $L(v)$.

Clearly, if (G, L) is a labeled graph, then $\text{cwd}(G) \leq \text{cwd}(G, L)$. Furthermore, if L is a constant labeling of a graph G,[23] then $\text{cwd}(G, L) = \text{cwd}(G)$.

Lemma 3.1. *Let G be a complete graph, and let $L : V(G) \to C$ be a labeling of G. Then $cwd(G, L) \leq |C| + 1$.*

Proof. Let $n = |V(G)|$ and $k = |C|$. We set $V(G) = \{v_1, \dots, v_n\}$, and clearly, we may assume that $C = \{1, \dots, k\}$. We construct (G, L) using labels $0, 1, \dots, k$ as follows. We first create vertex v_1 with label $L(v_1)$. If $n = 1$, then we have created (G, L), and we are done. Otherwise, we proceed inductively as follows. For each $i \in \{1, \dots, n-1\}$, having created the labeled graph $(G[v_1, \dots, v_i], L)$ using only labels $0, 1, \dots, k$, we create a new vertex v_{i+1} with label 0, we then make all vertices with label 0 (note that v_{i+1} is the only vertex with this label, since $0 \notin C$) adjacent to all vertices with labels $1, \dots, k$, and finally, we rename label 0 to label $L(v_{i+1})$. We have now created the labeled graph $(G[v_1, \dots, v_{i+1}], L)$ using only labels $0, 1, \dots, k$. This completes the induction. $\square$

Lemma 3.2. *Let G_1, G_2 be graphs on disjoint vertex sets, and let $L_1 : V(G_1) \to C_1$ and $L_2 : V(G_2) \to C_2$ be labelings of G_1 and G_2, respectively. Let (G, L) be the disjoint union of (G_1, L_1) and (G_2, L_2). Then $cwd(G) \leq \max\{cwd(G_1, L_1), cwd(G_2, L_2), |C_1 \cup C_2|\}$.*

[21]As usual, for a function $f : A \to B$ and a set $A' \subseteq A$, we denote by $f \upharpoonright A'$ the restriction of f to A'.

[22]Note that these are the same four operations that we had in the definition of the clique-width of nonlabeled graphs. The only difference is that, here, we insist that the labeling of G at the end of the procedure be precisely the labeling L.

[23]This simply means that L is a constant function with domain $V(G)$, that is, that L assigns the same label to all vertices of G.

Proof. Set $k = \max\{\text{cwd}(G_1, L_1), \text{cwd}(G_2, L_2), |C_1 \cup C_2|\}$. Since $|C_1 \cup C_2| \leq k$, we may assume that $C_1, C_2 \subseteq \{1, \dots, k\}$. We now (separately) construct (G_1, L_1) and (G_2, L_2) using only labels $1, \dots, k$, and then we take the disjoint union of the two labeled graphs. We have now constructed (G, L) using only labels $1, \dots, k$, and the result follows. □

A *3-peaked labeled graph* is a labeled graph (G, L) such that $V(G)$ can be partitioned into three (possibly empty) cliques, call them X, Y, Z,[24] such that all the following hold:

- X is anticomplete to Z;
- if $Y \neq \emptyset$, then Y can be ordered as $Y = \{y_1, \dots, y_t\}$ so that $N_G[y_t] \subseteq \dots \subseteq N_G[y_1]$;
- there exist three pairwise distinct labels, call them ℓ_1, ℓ_2, ℓ_3, such that L assigns label ℓ_1 to all vertices of X, label ℓ_2 to all vertices of Y, and label ℓ_3 to all vertices of Z.

Under these circumstances, we say that (X, Y, Z) is a *3-peaked partition* of the 3-peaked labeled graph (G, L).[25]

Lemma 3.3. *Every 3-peaked labeled graph (G, L) satisfies cwd$(G, L) \leq 5$.*

Proof. We proceed by induction on $|Y|$. More precisely, we fix a 3-peaked labeled graph (G, L) with 3-peaked partition (X, Y, Z), and we assume inductively that for every 3-peaked labeled graph (G', L') with 3-peaked partition (X', Y', Z'), if $|Y'| < |Y|$, then $\text{cwd}(G', L') \leq 5$. We must show that $\text{cwd}(G, L) \leq 5$. We may assume that L assigns label 1 to all vertices of X, label 2 to all vertices of Y, and label 3 to all vertices of Z.

Suppose first that Y is complete to $X \cup Z$. If at least one of X, Y, Z is empty, then either

- (G, L) is a complete labeled graph, and the labeling L uses at most two labels,[26] or
- (G, L) is the disjoint union of two complete labeled graphs, each with a constant labeling.[27]

In either case, Lemmas 3.1 and 3.2 imply that $\text{cwd}(G, L) \leq 3$, and we are done. So we may assume that X, Y, Z are all nonempty. Then $(G[X \cup Z], L)$ is the disjoint union of two complete labeled graphs, namely $(G[X], L)$ and

[24]Since our graphs are nonnull, at least one of X, Y, Z is nonempty.

[25]Note that the definition of a 3-peaked graph in fact implies that if $X \neq \emptyset$, then X can be ordered as $X = \{x_1, \dots, x_s\}$ so that $N_G[x_s] \subseteq \dots \subseteq N_G[x_1]$. A similar statement holds for Z.

[26]This happens if X or Z is empty.

[27]This happens if $Y = \emptyset$ and $X, Z \neq \emptyset$.

$(G[Z], L)$, each with a constant labeling, and so Lemmas 3.1 and 3.2 imply that $\mathrm{cwd}(G[X \cup Z], L) \leq 2$. On the other hand, $(G[Y], L)$ is a complete graph with a constant labeling, and so by Lemma 3.1, we have that $\mathrm{cwd}(G[X \cup Y], L) \leq 2$. Next, by Lemma 3.2, the disjoint union of labeled graphs $(G[X \cup Z], L)$ and $(G[Y], L)$ has clique-width at most three. Finally, we can turn this disjoint union into our labeled graph (G, L) by making all vertices with label 2 (*i.e.* all vertices in Y) adjacent to all vertices with labels $1, 3$ (*i.e.* to all vertices in $X \cup Z$), and we deduce that $\mathrm{cwd}(G, L) \leq 3$.

From now on, we assume that Y is not complete to $X \cup Z$. In particular, $Y \neq \emptyset$. Set $Y = \{y_1, \dots, y_t\}$ so that $N_G[y_t] \subseteq \dots \subseteq N_G[y_1]$, as in the definition of a 3-peaked labeled graph. Set $X_t = N_G(y_t) \cap X$ and $Z_t = N_G(y_t) \cap Z$. Since y_t is dominated by all other vertices of Y in G, we see that Y is complete to $Y_t \cup Z_t$; since Y is not complete to $X \cup Z$, we deduce that at least one of $X \setminus X_t$ and $Z \setminus Z_t$ is nonempty. Set $G' = G \setminus (X_t \cup \{y_t\} \cup Z_t)$. Clearly, (G', L) is a 3-peaked graph with 3-peaked partition $(X \setminus X_t, Y \setminus \{y_t\}, Z \setminus Z_t)$, and so by the induction hypothesis, we have that $\mathrm{cwd}(G', L) \leq 5$. Now, by repeatedly applying Lemmas 3.1 and 3.2, we see that (G, L) can be constructed using only labels $1, \dots, 5$, as follows. We first create the disjoint union of (G', L) and of the complete graph $G[X_t \cup \{y_t\}]$, with all vertices in X_t labeled 4 and the vertex y_t labeled 5. We then make all vertices labeled 4 (*i.e.* all vertices in X_t) adjacent to all vertices labeled $1, 2$ (*i.e.* to all vertices in $(X \setminus X_t) \cup (Y \setminus \{y_t\})$), and we make all vertices labeled 5 (note that y_t is the only such vertex) adjacent to all vertices labeled 2 (*i.e.* to all vertices in $Y \setminus \{y_t\}$). We rename label 4 as 1, and we rename label 5 as 2. We have now created the labeled graph $(G \setminus Z_t, L)$ using only labels $1, \dots, 5$. If $Z_t = \emptyset$, then we are done. So assume that $Z_t \neq \emptyset$. Then we take the disjoint union of the labeled graph $(G \setminus Z_t, L)$ and the complete graph $G[Z_t]$ with all vertices in Z_t labeled 4. Finally, we make all vertices labeled 4 (*i.e.* all vertices in Z_t) adjacent to all vertices labeled $2, 3$ (*i.e.* all vertices in $Y \cup (Z \setminus Z_t)$), and we rename label 4 as 3. We have now created the labeled graph (G, L) using only labels $1, \dots, 5$. This completes the argument. $\square$

We remind the reader that "5-baskets" and "5-crowns" were defined in Section 2, and that these graphs appear in our decomposition theorem for $(4K_1, C_4, C_6, C_7)$-free graphs (see Theorem 2.2). We now prove that 5-baskets and 5-crowns have bounded clique-width.

Lemma 3.4. *Every 5-basket Q satisfies $\mathrm{cwd}(Q) \leq 5$.*

Proof. Let Q be a 5-basket, and let $(A; B_1, B_2, B_3; C_1, C_2, C_3; F)$ be an associated 5-basket partition of Q. Let $H = Q[A \cup B_1 \cup B_2 \cup B_3]$, and let

$L_H : V(H) \to \{0, 1, 2, 3\}$ be given by

$$L_H(v) = \begin{cases} 0 & \text{if} \quad v \in A \\ 1 & \text{if} \quad v \in B_1 \\ 2 & \text{if} \quad v \in B_2 \\ 3 & \text{if} \quad v \in B_3 \end{cases}$$

for all $v \in V(H)$.

Claim 1. *cwd*$(H, L_H) \leq 5$.

Proof of Claim 1. By the definition of a 5-basket, and by symmetry, we may assume that A is complete to $B_2 \cup B_3$. Now $(H[A \cup B_1 \cup B_2], L_H)$ is a 3-peaked labeled graph with 3-peaked partition (B_1, A, B_2), and so by Lemma 3.3, $\text{cwd}(H[A \cup B_1 \cup B_2], L_H) \leq 5$. On the other hand, Lemma 3.1 guarantees that $\text{cwd}(H[B_3], L_H) \leq 2$. We then take the disjoint union of the labeled graphs $(H[A \cup B_1 \cup B_2], L_H)$ and $(H[B_3], L_H)$, and we note that, by Lemma 3.2, the resulting labeled graph has clique-width at most five. Finally, we make all vertices labeled 0 (*i.e.* all vertices in A) adjacent to all vertices labeled 3 (*i.e.* to all vertices in B_3), and we thus obtain the labeled graph (H, L_H). This proves Claim 1. ■

By the definition of a 5-basket, and by symmetry, we may assume that F is complete to $A \cup B_1 \cup B_2 \cup C_1 \cup C_2$ and anticomplete to $B_3 \cup C_3$. By Claim 1, we have that $\text{cwd}(H, L_H) \leq 5$. Now, by repeatedly applying Lemmas 3.1 and 3.2, we see that Q can be constructed using only labels $0, 1, 2, 3, 4$, as follows. First, we take the disjoint union of (H, L_H) and the complete graph $Q[C_1]$, with all vertices of C_1 labeled 4, and then we make all vertices labeled 1 (*i.e.* all vertices in B_1) adjacent to all vertices labeled 4 (*i.e.* to all vertices in C_1). Then, we rename label 1 as 0. Next, we take the disjoint union of the resulting labeled graph and the complete graph $Q[C_2]$, with all vertices of C_2 labeled 1, and then we make all vertices labeled 1 (*i.e.* all vertices in C_2) adjacent to all vertices labeled $2, 4$ (*i.e.* to all vertices in $B_2 \cup C_1$). Then, we rename label 2 as 0, and we rename label 1 as 4. Next, we create the disjoint union of the resulting labeled graph with the complete graph $Q[C_3]$, with all vertices in C_3 labeled 1, and then we make all vertices labeled 1 (*i.e.* all vertices in C_3) adjacent to all vertices labeled $3, 4$ (*i.e.* to all vertices in $B_3 \cup C_1 \cup C_2$). If $F = \emptyset$, then we have already created the graph Q, and we are done. So assume that $F \neq \emptyset$. We now take the disjoint union of the labeled graph that we just created, and of the complete graph $Q[F]$, with all vertices of F labeled 2. Finally, we make all vertices labeled 2 (*i.e.* all vertices in F) adjacent to all vertices labeled $0, 4$ (*i.e.* to all vertices in $A \cup B_1 \cup B_2 \cup C_1 \cup C_2$). We have now constructed the graph Q using only labels $0, 1, 2, 3, 4$. This proves that $\text{cwd}(Q) \leq 5$, which is what we needed to show. □

Lemma 3.5. *Every 5-crown Q satisfies cwd$(Q) \leq 5$.*

Proof. Let Q be a 5-crown with 5-crown partition $(X_0, X_1, X_2, X_3, X_4)$. By the definition of a 5-crown, and by symmetry, we may assume that X_1 is complete to X_2, and X_3 is complete to X_4. Let $L : V(Q) \to \{0, 1, 2, 3, 4\}$ be such that for all $i \in \{0, 1, 2, 3, 4\}$, L assigns label i to all vertices of X_i. Then $(Q[X_4, X_0, X_1], L)$ is a 3-peaked graph with 3-peaked partition (X_4, X_0, X_1), and $(Q[X_2, X_3], L)$ is a 3-peaked graph with 3-peaked partition $(X_2, X_3, \emptyset)$. So, Lemma 3.3 implies that $\text{cwd}(Q[X_4 \cup X_0 \cup X_1], L) \leq 5$ and $\text{cwd}(Q[X_2 \cup X_3], L) \leq 5$. We now take the disjoint union of these two 3-peaked graphs; by Lemma 3.2, the resulting labeled graph has clique-width at most five. Finally, we make all vertices with label 1 (*i.e.* all vertices in X_1) adjacent to all vertices with label 2 (*i.e.* to all vertices in X_2), and we make all vertices with label 3 (*i.e.* all vertices in X_3) adjacent to all vertices with label 4 (*i.e.* to all vertices in X_4). We have now created the labeled graph (Q, L) using only five labels, and we deduce that $\text{cwd}(Q) \leq \text{cwd}(Q, L) \leq 5$. This completes the argument. □

We are now ready to prove Theorem 1.3, restated below for the reader's convenience.

Theorem 1.3. *Let G be a $(4K_1, C_4, C_6, C_7)$-free graph. Then either G has a simplicial vertex, or G satisfies cwd$(G) \leq 5$.*

Proof. We may assume that G has no simplicial vertices, for otherwise we are done. So, by Theorem 2.2, G has exactly one nontrivial anticomponent, call it Q, and this anticomponent is either a 5-basket or a 5-crown. By Lemmas 3.4 and 3.5, we have that $\text{cwd}(Q) \leq 5$. Let $K = V(G) \setminus V(Q)$; then K is a (possibly empty) clique, complete to $V(Q)$ in G. If $K = \emptyset$, then $G = Q$, and we are done. So assume that $K \neq \emptyset$.

Let $L_Q : V(Q) \to \{1\}$ be a constant labeling of Q, and let $L_K : K \to \{2\}$ be a constant labeling of the complete graph $G[K]$. Then $\text{cwd}(Q, L_Q) = \text{cwd}(Q) \leq 5$, and by Lemma 3.1, we have that $\text{cwd}(G[K], L_K) \leq 2$. We now take the disjoint union of (Q, L_Q) and $(G[K], L_K)$; by Lemma 3.2, the clique-width of the resulting graph is at most five. Finally, we make all vertices labeled 1 (*i.e.* all vertices in $V(Q)$) adjacent to all vertices labeled 2 (*i.e.* to all vertices of K); this produces the graph G, and it establishes that $\text{cwd}(G) \leq 5$. □

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