# On the clique-width of $(4K_1, C_4, C_5, C_7)$-free graphs

Irena Penev*

July 2, 2020

**Abstract**

We prove that $(4K_1, C_4, C_5, C_7)$-free graphs that are not chordal have unbounded clique-width. This disproves a conjecture from [D.J. Fraser, A.M. Hamel, C.T. Hoàng, K. Holmes, T.P. LaMantia, *Characterizations of* $(4K_1, C_4, C_5)$*-free graphs*, Discrete Applied Mathematics, 231:166–174, 2017].

Brandstädt *et al.* [1] constructed a family of $4K_1$-free chordal graphs of unbounded clique-width. In the present note, we slightly modify their construction to show that $(4K_1, C_4, C_5, C_7)$-free graphs that are not chordal have unbounded clique-width (see Theorem 4). This disproves a conjecture due to Fraser *et al.* [3].

We begin with a few definitions. In what follows, all graphs are finite, simple, and nonnull. For a graph $H$, a graph $G$ is said to be *$H$-free* if no induced subgraph of $G$ is isomorphic to $H$. For a family of graphs $\mathcal{H}$, $G$ is said to be *$\mathcal{H}$-free* if $G$ is $H$-free for all $H \in \mathcal{H}$. As usual, $nK_1$ ($n \geq 1$) is the edgeless graph on $n$ vertices, and $C_n$ ($n \geq 3$) is the cycle on $n$ vertices. A *hole* is an induced cycle of length at least four; a graph is *chordal* if it contains no holes. The vertex set of a graph $G$ is denoted by $V(G)$, and the complement of $G$ is denoted by $\overline{G}$. For a nonempty set $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of $G$ induced by $X$. For a set $X \subsetneqq V(G)$, $G \setminus X$ is the subgraph of $G$ obtained by deleting all vertices in $X$, *i.e.* $G \setminus X = G[V(G) \setminus X]$. A *clique* in $G$ is a (possibly empty) set of pairwise adjacent vertices of $G$, and a *stable set* in $G$ is a (possibly empty) set of pairwise nonadjacent vertices of $G$. For a vertex $x$ of $G$, $N_G(x)$ the set of all neighbors of $x$ in $G$, and $N_G[x] = \{x\} \cup N_G(x)$; $x$ is *simplicial* in $G$ if $N_G(x)$ is a clique in $G$. It is well known (see [4]) that every chordal graph contains a simplicial vertex.

Here, we are interested in the class of $(4K_1, C_4, C_5, C_7)$-free graphs. Since holes of length at least eight contain an induced $4K_1$, we see that $(4K_1, C_4, C_5, C_7)$-free graphs are precisely the $4K_1$-free graphs in which all holes are of length six.

The *clique-width* of a graph $G$, denoted by $\text{cwd}(G)$, is the minimum number of labels needed to construct $G$ using the following four operations:

1. creation of a new vertex $v$ with label $i$;
2. disjoint union of two labeled graphs;
3. joining by an edge every vertex labeled $i$ to every vertex labeled $j$;
4. renaming label $i$ to label $j$.

*Computer Science Institute of Charles University (IÚUK), Prague, Czech Republic. Email: `ipenev@iuuk.mff.cuni.cz`. Partially supported by project `17-04611S (Ramsey-like aspects of graph coloring)` of the Czech Science Foundation, and by Charles University project `UNCE/SCI/004`.

It is clear that if $H$ is an induced subgraph of $G$, then $\text{cwd}(H) \leq \text{cwd}(G)$. Further, the following is Theorem 4.1 from [2].

**Theorem 1.** *[2] For every graph $G$, we have $cwd(\overline{G}) \leq 2cwd(G)$.*

Brandstädt *et al.* [1] showed that $4K_1$-free chordal graphs have unbounded clique-width. On the other hand, Fraser *et al.* [3] showed that every $(4K_1, C_4, C_5, C_7)$-free graph either contains a simplicial vertex or has bounded clique-width, and furthermore, they conjectured the following.

**Conjecture 2.** *[3] Let $G$ be a $(4K_1, C_4, C_5, C_7)$-free graph. Then $G$ is chordal or has bounded clique-width.*

In this note, we adapt the construction from [1] to disprove Conjecture 2 (see Theorem 4).

We first need a few more definitions. Given a graph $G$ and disjoint sets $X, Y \subseteq V(G)$, we say that $X$ is *complete* (resp. *anticomplete*) to $Y$ if every vertex in $X$ is adjacent (resp. non-adjacent) to every vertex in $Y$. Given distinct vertices $x, y \in V(G)$, we say that $x$ *dominates* $y$ in $G$ if $N_G[y] \subseteq N_G[x]$. In particular, if one vertex dominates another, then the two vertices are adjacent. A *dominating vertex* in $G$ is a vertex that dominates all other vertices in $G$, or equivalently, a vertex that is adjacent to all other vertices of $G$.

Brandstädt *et al.* [1] constructed a family $\{G_n\}_{n=1}^{\infty}$ of graphs as follows. For an integer $n \geq 1$, $G_n$ is a graph on $n^2 + 2n$ vertices such that $V(G_n)$ can be partitioned into three sets, call them $A_n = \{a_1, \ldots, a_n\}$, $B_n = \{b_1, \ldots, b_n\}$, and $C_n = \{c_{i,j} \mid 1 \leq i, j \leq n\}$, with adjacency as follows:

- $A_n, B_n, C_n$ are stable sets;
- $A_n$ is complete to $B_n$;
- for all $i, i', j \in \{1, \ldots, n\}$ $a_i$ is adjacent to $c_{i',j}$ if and only if $i' \leq i$;
- for all $i, j, j' \in \{1, \ldots, n\}$, $b_j$ is adjacent to $c_{i,j'}$ if and only if $j' \leq j$.

The following is Lemma 12 from [1].

**Theorem 3.** *[1] For all integers $n \geq 1$, $cwd(G_n) \geq n$.*

We remark that $\{\overline{G_n}\}_{n=1}^{\infty}$ is the family of $4K_1$-free chordal graphs of unbounded clique-width from [1]. Indeed, by by Observation 1 from [1], graphs in this family are $4K_1$-free and chordal, and by Theorems 1 and 3, the family has unbounded clique-width.

We now construct a family $\{H_n\}_{n=1}^{\infty}$ of graphs by "attaching" a $C_6$ to the graphs $\overline{G_n}$ ($n \geq 1$) in a convenient way, *i.e.* in a way that does not introduce any of our four forbidden induced subgraphs. Formally, for an integer $n \geq 1$, $H_n$ is the graph whose vertex set can be partitioned into four sets, call them $A_n = \{a_1, \ldots, a_n\}$, $B_n = \{b_1, \ldots, b_n\}$, $C_n = \{c_{i,j} \mid 1 \leq i, j \leq n\}$, and $X = \{x_0, x_1, x_2, x_3, x_4, x_5\}$, with adjacency as follows:

- $A_n, B_n, C_n$ are cliques;
- $A_n$ is anticomplete to $B_n$;
- for all $i, i', j \in \{1, \ldots, n\}$ $a_i$ is adjacent to $c_{i',j}$ if and only if $i' \geq i + 1$;
- for all $i, j, j' \in \{1, \ldots, n\}$, $b_j$ is adjacent to $c_{i,j'}$ if and only if $j' \geq j + 1$;
- $x_0, x_1, x_2, x_3, x_4, x_5, x_0$ is a hole of length six;

- $A_n$ is complete to $\{x_0, x_1\}$ and anticomplete to $\{x_2, x_2, x_3, x_4\}$;
- $B_n$ is complete to $\{x_2, x_3\}$ and anticomplete to $\{x_4, x_5, x_0, x_1\}$;
- $C_n$ is complete to $X$.

Note that $H_n \setminus X = \overline{G_n}$. The following theorem shows that Conjecture 2 is false.

**Theorem 4.** *For all integers $n \geq 1$, $H_n$ is $(4K_1, C_4, C_5, C_7)$-free and not chordal, and it satisfies cwd$(H_n) \geq \frac{n}{2}$.*

*Proof.* Fix an integer $n \geq 1$, and let $A_n$, $B_n$, $C_n$, and $X$ be as in the definition of $H_n$. Then $H_n \setminus X = \overline{G_n}$, and so Theorems 1 and 3 imply that $\text{cwd}(H_n) \geq \frac{n}{2}$. Further, by construction, $x_0, x_1, x_2, x_3, x_4, x_5, x_0$ is a hole of length six in $H_n$, and so $H_n$ is not chordal.

Since $(A_n \cup \{x_0, x_1\}, B_n \cup \{x_2, x_3\}, C_n \cup \{x_4, x_5\})$ is a partition of $V(H_n)$ into three cliques, $H_n$ is $4K_1$-free. It remains to show that $H_n$ is $(C_4, C_5, C_7)$-free. First, note that for any two distinct vertices in $A_n$, one of the two vertices dominates the other in $H_n$. Since no vertex in a hole dominates any other vertex in that hole, we see that a hole in $H_n$ can contain at most one vertex of $A_n$. But for all $i \in \{1, \dots, n\}$, $N_{H_n}(a_i) \setminus A_n \subseteq C_n \cup \{x_0, x_1\}$, and $C_n \cup \{x_0, x_1\}$ is a clique in $H_n$. Since holes contain no simplicial vertices, we see that no hole in $H_n$ intersects $A_n$. Similarly, no hole in $H_n$ intersects $B_n$. Finally, every vertex in $C_n$ is dominating in $H_n \setminus (A_n \cup B_n)$; since holes contain no dominating vertices, it follows that no hole in $H_n$ intersects $C_n$. Thus, every hole in $H_n$ is in fact a hole in $H_n[X]$, and we deduce that $x_0, x_1, x_2, x_3, x_4, x_5, x_0$ is the only hole in $H_n$. Since this hole is of length six, we see that $H_n$ is $(C_4, C_5, C_7)$-free. □

# References


[1] A. Brandstädt, J. Engelfriet, H.O.Le, V.V. Lozin, *Clique-width for 4-vertex forbidden subgraphs*, Theory of Computing Systems 39:561–590, 2006.

[2] B. Courcelle, S. Olariu, *Upper bounds to the clique width of graphs*, Discrete Applied Mathematics, 101:77–114, 2000.

[3] D.J. Fraser, A.M. Hamel, C.T. Hoàng, K. Holmes, T.P. LaMantia, *Characterizations of $(4K_1, C_4, C_5)$-free graphs*, Discrete Applied Mathematics, 231:166–174, 2017.

[4] D.R. Fulkerson, O.A. Gross, *Incidence matrices and interval graphs*, Pacific Journal of Mathematics, 15:835–855, 1965.