Mathematics++

Practicals 4 – Measure concentration

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1. Given $t \in [-1, 1]$, let $C(t)$ be the spherical cap of height $1 - t$. That is,

$$C(t) := \left\{ x \in S^{n-1} : x_1 \geq t \right\}.$$

Prove that $\mu(C(t)) \leq e^{-t^2 n/2}$, where $\mu(C(t)) := \lambda(\tilde{C}(t))/\lambda(B^n)$ and $\tilde{C}(t)$ is the cone over $C(t)$ with apex in the origin.
Hint: Show that $\tilde{C}(t)$ is a subset of a suitable ball. (For small t try to get radius $\sqrt{1-t^2}$, for larger t, try to get another radius.)

2. Deduce the measure concentration for the dot product of two random unit vectors in $\mathbb{R}^n$.

3. Estimate the probability that a random line passing through origin intersects the unit ball centered at $(2, 0, 0, \ldots, 0)$.

4. Using Lévy's lemma, show that the expected value $\int_{S^{n-1}} f(x) \, \mathrm{d}\mu(x)$ of a 1-Lipschitz function $f \colon S^{n-1} \to \mathbb{R}$ has distance at most $O(1/\sqrt{n})$ from the median.