Mathematics++

Practicals 2 – Measure and Integral

March 20, 2019

1. Show that for every measurable $A \subseteq \mathbb{R}^k$ there exist Borel sets $B, C \in \mathbb{R}^k$ such that $B \subseteq A \subseteq C$ and $\lambda(A \setminus B) = \lambda(C \setminus A) = 0$. (Which means that every measurable set can be approximated by Borel sets with 0 error both from inside and from outside.)

2. Let $(X, \mathcal{S}, \mu)$ be a measurable space and let $f, g : X \to \mathbb{R}$ be simple non-negative functions. Show that value of the integral does not depend on the way we write the simple function and therefore
$$\int (f+g)\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu + \int g\,\mathrm{d}\mu.$$

3. Let $(X, \mathcal{S}, \mu)$ be a measurable space and let $f : X \to \mathbb{R}$ be a measurable function. Show that for every $X, X' \subseteq Y$ such that $\mu(X \,\Delta\, X') = 0$ holds
$$\int_X f\,\mathrm{d}\mu = \int_{X'} f\,\mathrm{d}\mu$$
given that at least one of the integrals is defined.

4. Prove Monotone convergence theorem: Let $(X, \mathcal{S}, \mu)$ be a measurable space and let $f, f_n : X \to \mathbb{R}$ be non-negative measurable functions. If $f_n \to f$ almost everywhere on $D \subseteq X$ and $f_n \leq f$ then $\int_D f = \lim_n \int_D f_n$. *Hint:* Use Fatou's lemma.

5. Find a sequence of continuous functions $f_n : [0,1] \to [0,\infty)$ such that:
   - $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$,
   - $\int_0^1 f_n(x)dx \to 0$ for $n \to \infty$,
   - $\sup_{n\in\mathbb{N}} f_n$ is not Lebesgue integrable.

6. Calculate the following integral
$$\int_0^1 \frac{\log(1-x)}{x}\,\mathrm{d}x$$
Hint: Use Taylor expansion of a suitable function. [*]

7. Design a suitable probability space for experiment "choose 3 points in a unit square (uniformly and independently)".