

Mathematics++

Problem set 3 – Convex geometry

hints after **April 30, 2019**, solutions due **May 7, 2019**

Prékopa-Leindler inequality: Let $t \in (0, 1)$ and let $f, g, h: \mathbb{R}^n \rightarrow \mathbb{R}$ be measurable, non-negative, bounded functions with a finite integral over \mathbb{R}^n . Suppose that $h((1-t)x + ty) \geq f(x)^{1-t}g(y)^t$ for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f \right)^{1-t} \left(\int_{\mathbb{R}^n} g \right)^t.$$

Log-concavity: A non-negative function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is called log-concave if it satisfies

$$f((1-t)x + ty) \geq f(x)^{1-t}f(y)^t \text{ for all } t \in (0, 1), x, y \in \mathbb{R}^n.$$

Edge expansion: Edge expansion of a graph $G = (V, E)$ is

$$\min \left\{ \frac{e(A, V \setminus A)}{|A|} : A \subseteq V, 1 \leq |A| \leq \frac{1}{2}|V| \right\}$$

where $e(A, B)$ is the number of edges of G going between A and B .

1. Prove the Prékopa-Leindler inequality using the Prékopa-Leindler inequality with the additional assumption that $\sup h = \sup f = \sup g = 1$. [3]

2. Show that a positive function $f(x)$ is log-concave if and only if the function $\log(f(x))$ is concave. [2]

3. Show that log-concave functions are closed under products, projections and convolutions.

The projection of a function $f: \mathbb{R}^{m+n} \rightarrow \mathbb{R}$ is a function $g: \mathbb{R}^m \rightarrow \mathbb{R}$ defined as $g(x) := \int_{\mathbb{R}^n} f(x, y) dy$, and the convolution is $(f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y) dy$ for $f, g: \mathbb{R}^n \rightarrow \mathbb{R}$. [6]

4. Let \mathcal{G} be an (infinite) class of graphs such that its members have maximum degree at most d and edge expansion at least $c_0 > 0$. Show that there exists $c > 0$ such that for all $G \in \mathcal{G}$ the following holds

$$1 - \frac{|A_t|}{|V(G)|} \leq e^{-ct}$$

for every $A \subseteq V(G)$ such that $|A| \geq \frac{1}{2}|V(G)|$ and $t \geq 0$. A_t is the set of all vertices with distance at most t from some element of A (in particular $A_t \supseteq A$). [4]

5. Let $A, B \subseteq \mathbb{R}^d$ be convex sets. Prove that: [3]

$$\text{conv}(\{\{0\} \times A\} \cup \{\{1\} \times B\}) = \bigcup_{t \in [0, 1]} (\{t\} \times ((1-t)A \oplus tB))$$