Mathematics++

Problem set 2 – Measure and integral

hints after **April 2, 2019**, solutions due **April 9, 2019**

1. Let $(X, \mathcal{S}, \mu)$ be a measurable space and $f, g : X \to \mathbb{R}$ be measurable functions. Prove that if $\int g < \infty$ a $|f| \leq g$ almost everywhere, then $\int f < \infty$. **[3]**

2. (a) There is a function $f : [0,1] \to \mathbb{R}$ such that the Lebesgue integral $\int_0^1 f$ is finite (in particular exists) but the Newton integral $\int_0^1 f$ does not exist? **[2]**

   (b) There is a function $f : [0,\infty) \to \mathbb{R}$ such that the Newton integral $\int_0^\infty f$ is finite (in particular exists) but the Lebesgue integral $\int_0^\infty f$ does not exist? **[3]**

3. Construct a sequence of continuous function $f_n\colon [0,1] \to [0,1]$ such that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$ but the sequence $\{f_n(x)\}$ does not converge for any $x \in [0,1]$. **[4*]**

4. Show that the function $f\colon (0,1)\times(0,1) \to \mathbb{R}$ defined as

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

   is not (Lebesgue) integrable on $(0,1)\times(0,1)$. **[4*]**

5. Let $f\colon \mathbb{R}^k \to \mathbb{R}$ be bounded Lebesgue mesuarable function. Prove that there are Borel functions $g, h\colon \mathbb{R}^k \to \mathbb{R}$ such that $g = h$ almost everywhere and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}^k$. **[4]**

6. Determine the volume of the unit ball B_n according to the following steps.

   (a) Let $I_n = \int_{\mathbb{R}^n} e^{-\|x\|^2/2}\,\mathrm{d}x$ where $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ is the Euclidean norm. Express I_n using I_1. **[1]**

   (b) Express I_n using $V_n = \mathrm{Vol}(B_n)$ and some 1-dimensional integral: Consider the increment of I_n on the spherical shell of the inner radius r and the outer radius $r + \mathrm{d}r$ (that is the set $\{x \in \mathbb{R}^n\colon \|x\| \in [r, r+\mathrm{d}r]\}$). **[2*]**

   (c) Determine V_n using (b). **[2*]**
   Hint: Recall how to deduce $n! = \int_0^\infty x^n e^{-x}\,\mathrm{d}x$.