Mathematics++

Problem set 1 – Measure and σ-algebras

hints after **19 March 2019**, solutions due **26 March 2019**

Definition: Let $x \in \mathbb{R}$ and $E \subset \mathbb{R}$ be a measurable set. We define the **density** of E at x as the limit

$$d_E(x) := \lim_{\delta \to 0} \frac{\lambda((x-\delta, x+\delta) \cap E)}{2\delta}$$

if such limit exists.

1. Prove or disprove that there exists an infinite σ-algebra with countably many elements only. **[5*]**

2. Let $(X, \mathcal{S}, \mu)$ be a measurable space and $\{A_i\}_{i=0}^{\infty}$ be a sequence of measurable sets such that $A_{i+1} \subseteq A_i$ for every i. Assuming $\mu(A_0) < \infty$, show that

   $$\lim_{i \to \infty} \mu(A_i) = \mu\left(\bigcap_{i=0}^{\infty} A_i\right).$$

   In addition, show that the assumption above is necessary, that is, find a sequence as above which violates $\mu(A_0) < \infty$ as well as the conclusion. **[4]**

3. Prove that the set C defined below is mesuarable and determine its (Lebesgue) measure.

   Let $\{\mathcal{K}_n\}$ be a sequence of finite collections of closed intervals defined inductively as

   - $\mathcal{K}_0 = \{[0,1]\}, \mathcal{K}_1 = \{[0, \frac{1}{3}], [\frac{2}{3}, 1]\}$,
   - $\mathcal{K}_n$ is obtained from $\mathcal{K}_{n-1}$ by removing the open middle third of each of the intervals in $\mathcal{K}_{n-1}$.

   Then, we set $K_n := \bigcup \mathcal{K}_n$ and $C := \bigcap_n K_n$. **[4]**

4. Show that every measurable set of finite measure can be approximated with arbitrary precision by a finite union of intervals; that is, $\forall E \subset \mathbb{R}$ of finite measure and $\forall \varepsilon > 0$ there is $A \subset \mathbb{R}$, which is a union of finitely many open intervals, such that $\lambda(E \triangle A) \leq \varepsilon$.

   In addition, show that the assumption on finite measure is necessary; that is, find a measurable set of infinite measure which cannot be approximated by a finite union of intervals for some $\varepsilon > 0$. **[4]**

5. Show that every measurable subset of $\mathbb{R}$ of finite measure can be approximated from inside with arbitrary precision by a compact set (that is closed and bounded set). That is, show that $\forall E \subset \mathbb{R}$ of finite measure and $\forall \varepsilon > 0$ there is $K \subseteq E$ compact such that $\mu(E \setminus K) \leq \varepsilon$ **[4]**
   Hint: Try to solve the problem for bounded E first.

6. Prove that there is no measurable set $E \subseteq (0,1)$ such that $\lambda(E) = 1/2$ and for every $x \in (0,1)$ the density of E at x equals $1/2$. **[6*]**
   Hint: You may want to use the approximation from one of the problems above.

Webpage `http://iuuk.mff.cuni.cz/~husek/vyuka/m++1819.html`