NMAI057 – Linear algebra 1

Tutorial 12

Linear maps – isomorphisms

Date: January 4, 2022 TA: Pavel Hubáček

Problem 1. Decide and justify whether the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$f(x,y,z) = (x+y-2z, y-z, x-y)^T$$

is in isomorphism of $\mathbb{R}^3$ onto itself (so-called automorphism).

Problem 2. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by the images of basis B:

$$\begin{aligned} f(2,1,1) &= (1,2,3)^T, \\ f(1,3,5) &= (3,2,1)^T, \\ f(7,1,4) &= (1,1,1)^T. \end{aligned}$$

Decide and justify whether:

(a) f is injective – if not then find distinct vectors $u, v \in \mathbb{R}^3$ such that $f(u) = f(v)$,

(b) f is surjective (onto) – if not then find a vector without a preimage, i.e., $u \in \mathbb{R}^3$ such that for all $v \in \mathbb{R}^3$ it holds that $f(v) \neq u$.

Compute the dimension and find a basis for both the image and kernel of f.

Problem 3. Let $f\colon U \to V$ and $g\colon V \to W$ be isomorphisms. Prove that their composition $g \circ f\colon U \to W$ is also an isomorphism. In particular, show that:

(a) $g \circ f$ is injective,

(b) $g \circ f$ is surjective.

Problem 4. Decide and justify whether the following vector spaces are isomorphic:

(a) $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^4$,

(b) $\mathbb{R}^4$ and $\mathcal{P}^3$ (the space of all real polynomials of degree at most three),

(c) $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{n\times m}$,

(d) $\mathbb{R}^n$ over $\mathbb{R}$ and $\mathbb{C}^n$ over $\mathbb{R}$,

(e) $\mathbb{R}^2$ and $\{v \in \mathbb{R}^4 \mid x_1 + x_2 = x_3 + x_4 = 0\}$,

(f) the space of all real polynomials and the space of all real sequences,

(g) $\mathbb{R}^4$ and the space of all linear maps (forms) $f\colon \mathbb{R}^4 \to \mathbb{R}$.

Problem 5. For the linear map $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined as $A \mapsto A - A^T$, decide and justify which of the given vectors are elements of the image of f and which are elements of the kernel of f:

(a)
$$I_2,$$

(b)
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

(c)
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

(d)
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Problem 6. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Denote $f^1 = f$, $f^2 = f \circ f, \ldots, f^n = f \circ f^{n-1}$. Prove that $\mathrm{Ker}(f^n) \subseteq \mathrm{Ker}(f^{n+1})$.

Problem 7. Decide and justify whether the given linear map is injective and surjective:

(a) $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^3$ defined as $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b+c, a+b, a)^T$,

(b) $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^4$ defined as $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b+c+d, a+b+c, a+b, a)^T$,

(c) $f\colon \mathcal{P}^2 \to \mathbb{R}^4$ defined as $f(ax^2+bx+c) = (a+b, 2b-c, a-b+c, a+b)^T$,

(d) $f\colon \mathcal{P}^2 \to \mathbb{R}^3$ defined as $f(ax^2+bx+c) = (a+b, 2b-c, a-b+c)^T$,

(e) $f\colon \mathcal{P}^2 \to \mathbb{R}^3$ defined as $f(ax^2+bx+c) = (a+b, 2b-c, a-b+2c)^T$.

Problem 8. Prove that for all $A \in \mathbb{R}^{n\times p}$, $B \in \mathbb{R}^{p\times n}$,

$$\dim(\mathrm{Ker}(A) \cap \mathcal{C}(B)) = \mathrm{rank}(B) - \mathrm{rank}(AB),$$

where $\mathcal{C}(B)$ denotes the column space of B.