NMAI057 – Linear algebra 1

Tutorial 10 & 11

Linear maps

Date: December 14 and 21, 2021 TA: Pavel Hubáček

Problem 1. Decide and justify whether the following real functions are linear maps

(a) $f_1(x) = 0$,
(b) $f_2(x) = 1$,
(c) $f_3(x) = 2x$,
(d) $f_4(x) = x + 1$,
(e) $f_5(x) = x^2$.

Problem 2. Decide and justify whether the following transformations of $\mathbb{R}^2$ are linear maps

(a) $f_6((x_1, x_2)^T) = (x_1 + x_2, x_1 - x_2)^T$,
(b) $f_7((x_1, x_2)^T) = (x_1 - x_2, x_1 - x_2)^T$.

Problem 3. For the transformation $f_6 : \mathbb{R}^2 \to \mathbb{R}^2$ defined above, find the matrix $[f_6]_{K_2,K_2}$ of f_6 w.r.t. the standard basis $K_2 = \{e_1 = (1, 0)^T, e_2 = (0, 1)^T\}$ of $\mathbb{R}^2$.

Problem 4. Consider the basis $B_1 = \{(-1, 0, 3)^T, (2, -2, 2)^T, (0, 1, -3)^T\}$ of $\mathbb{R}^3$. Find the matrix of $f : \mathbb{R}^3 \to \mathbb{R}^3$ w.r.t. the basis B_1 (i.e., $[f]_{B_1,B_1}$) if you know that f maps the basis vectors as follows (note that all vectors are scaled by a factor of 2):

$$f((-1, 0, 3)^T) = (-2, 0, 6)^T,$$
$$f((2, -2, 2)^T) = (4, -4, 4)^T,$$
$$f((0, 1, -3)^T) = (0, 2, -6)^T.$$

For x with coordinates $[x]_{B_1} = (1, 2, -1)^T$, use the matrix $[f]_{B_1,B_1}$ to compute the coordinates $[f(x)]_{B_1}$ of the image of x under f w.r.t. B_1.

Problem 5. For the linear map f from the previous problem, find the matrix $[f]_{B_1,B_2}$ of f w.r.t. the bases

$$B_1 = \{x_1 = (-1, 0, 3)^T, x_2 = (2, -2, 2)^T, x_3 = (0, 1, -3)^T\} \text{ and}$$
$$B_2 = \{y_1 = (-1, 1, 0)^T, y_2 = (0, 1, -1)^T, y_3 = (1, 0, 1)^T\}.$$

For x with coordinates $[x]_{B_1} = (1, 2, -1)^T$, use the matrix $[f]_{B_1,B_2}$ to compute the coordinates $[f(x)]_{B_2}$ of the image of x under f w.r.t. B_2.

Problem 6. For the bases B_1 and B_2 from the previous problem, find the change of basis matrix $[id]_{B_1,B_2}$ that transforms coordinates w.r.t. B_1 into coordinates w.r.t. B_2.

For x with coordinates $[x]_{B_1} = (1, 2, -1)^T$, use the change of basis matrix $[id]_{B_1,B_2}$ to compute the coordinates $[x]_{B_2}$ of x w.r.t. B_2.

Problem 7. How about transforming the coordinates $[x]_{B_2}$ of x w.r.t. B_2 into coordinates w.r.t. B_1? Find the change of basis matrix $[id]_{B_2,B_1}$ that transforms coordinates w.r.t. B_2 into coordinates w.r.t. B_1.

For x with coordinates $[x]_{B_2} = (1, -6, 4)^T$, use the matrix $[id]_{B_2,B_1}$ to compute the coordinates $[x]_{B_1}$ of x w.r.t. B_1.

Problem 8. Consider $f\colon \mathbb{Z}_5^3 \to \mathbb{Z}_5^3$ defined by the matrix

$$[f]_{B,K_3} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 4 & 0 & 3 \end{pmatrix}$$

w.r.t. the standard basis K_3 of $\mathbb{Z}_5^3$ and the basis $B = \{(3, 2, 1)^T, (1, 3, 4)^T, (2, 2, 2)^T\}$ of $\mathbb{Z}_5^3$.

Compute the matrix $[f]_{K_3,K_3}$ of f w.r.t. to the standard basis K_3 of $\mathbb{Z}_5^3$.

Problem 9. Consider $g\colon \mathbb{Z}_7^2 \to \mathbb{Z}_7^3$ defined by the matrix

$$[g]_{K_2,K_3} = \begin{pmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 6 \end{pmatrix}$$

w.r.t. to the standard bases K_2 of $\mathbb{Z}_7^2$ and K_3 of $\mathbb{Z}_7^3$.

Compute the matrix $[g]_{B_2,B_3}$ of g w.r.t. the bases $B_2 = \{(1, 4)^T, (3, 1)^T\}$ of $\mathbb{Z}_7^2$ and $B_3 = \{(1, 1, 2)^T, (1, 0, 3)^T, (6, 0, 5)^T\}$ of $\mathbb{Z}_7^3$.

Problem 10. Consider $h\colon \mathbb{Z}_5^2 \to \mathbb{Z}_5^3$ defined by the matrix

$$[h]_{B_2,B_3} = \begin{pmatrix} 4 & 3 \\ 2 & 4 \\ 3 & 1 \end{pmatrix}$$

w.r.t. the bases $B_2 = \{(4, 3)^T, (1, 4)^T\}$ of $\mathbb{Z}_5^2$ and $B_3 = \{(1, 1, 1)^T, (1, 4, 0)^T, (4, 0, 1)^T\}$ of $\mathbb{Z}_5^3$.

Compute the matrix $[h]_{K_2,K_3}$ of h w.r.t. the standard bases K_2 of $\mathbb{Z}_5^2$ and K_3 of $\mathbb{Z}_5^3$.

Problem 11. For the linear maps f and h defined above, compute the matrix $[f \circ h]_{K_2,K_3}$ of the composed map $f \circ h\colon \mathbb{Z}_5^2 \to \mathbb{Z}_5^3$ w.r.t. the standard bases K_2 of $\mathbb{Z}_5^2$ and K_3 of $\mathbb{Z}_5^3$.