NMAI057 – Linear algebra 1

Tutorial 10 & 11

Linear maps

Date: December 14 and 21, 2021 TA: Pavel Hubáček

Problem 1. Decide and justify whether the following real functions are linear maps

(a) $f_1(x) = 0$,
(b) $f_2(x) = 1$,
(c) $f_3(x) = 2x$,
(d) $f_4(x) = x + 1$,
(e) $f_5(x) = x^2$.

Solution:

Recall the definition of a linear map. For vector spaces U, V over a field $\mathbb{F}$, a map $f : U \to V$ is linear if for all $x, y \in U$ and $\alpha \in \mathbb{F}$:

(i) $f(x + y) = f(x) + f(y)$,
(ii) $f(\alpha x) = \alpha f(x)$.

We will verify the conditions from the definition for the given maps.

(a) For all $x, y \in \mathbb{R}$ a $\alpha \in \mathbb{R}$
  (i) $f_1(x + y) = 0 = 0 + 0 = f_1(x) + f_1(y)$ a
  (ii) $f_1(\alpha x) = 0 = \alpha 0 = \alpha f_1(x)$.

  Both conditions hold and f_1 is linear.

(b) Analogously for f_2:
  (i) The first condition is not satisfied since

$$f_2(x + y) = 1 \neq 2 = 1 + 1 = f_2(x) + f_2(y).$$

  (ii) There is no need to compute any further but we will check also the other condition

$$f_2(\alpha x) = f_2(w) = 1 \neq \alpha = \alpha 1 = \alpha f_2(x)$$

  and for all $\alpha \in \mathbb{R}$, neither the second condition holds.

  The map is not linear.

(c) For all $x, y \in \mathbb{R}$ a $\alpha \in \mathbb{R}$
   (i) $f_3(x+y) = 2(x+y) = 2(x) + 2(y) = f_3(x) + f_3(y)$ and
   (ii) $f_3(\alpha x) = 2\alpha x = \alpha 2x = \alpha f_3(x)$.

   Both conditions hold and the map is linear.

(d) It is a linear map. The check is similar to f_3 above.

(e) The map is not linear. Neither of the conditions hold. For example:
   (i) $f_5(x+y) = (x+y)^2 = x^2 + 2xy + y^2 \neq x^2 + y^2 = f_5(x) + f_5(y)$.

Problem 2. Decide and justify whether the following transformations of $\mathbb{R}^2$ are linear maps

(a) $f_6((x_1, x_2)^T) = (x_1 + x_2, x_1 - x_2)^T$,

(b) $f_7((x_1, x_2)^T) = (x_1 - x_2, x_1 - x_2)^T$.

Solution:

We proceed similarly to the above problem.

(a) The map f_6 is linear. For all $(x_1, x_2)^T, (y_1, y_2)^T \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$:
   (i) $f_6((x_1, x_2)^T + (y_1, y_2)^T) = f_6((x_1 + y_1, x_2 + y_2)^T) = (x_1 + y_1 + x_2 + y_2, x_1 + y_1 - x_2 - y_2)^T = (x_1 + x_2, x_1 - x_2)^T + (y_1 + y_2, y_1 - y_2)^T = f_6((x_1, x_2)^T) + f_6((y_1, y_2)^T)$ and
   (ii) $f_6(\alpha(x_1, x_2)^T) = f_6((\alpha x_1, \alpha x_2)^T) = (\alpha x_1 + \alpha x_2, \alpha x_1 - \alpha x_2)^T = \alpha(x_1 + x_2, x_1 - x_2)^T = \alpha f_6((x_1, x_2)^T)$.

(b) The map f_7 is linear. Analogous to f_6.

Note that we could have also used matrix representation of the maps and rely on properties of matrix product.

Problem 3. For the transformation $f_6 : \mathbb{R}^2 \to \mathbb{R}^2$ defined above, find the matrix $[f_6]_{K_2,K_2}$ of f_6 w.r.t. the standard basis $K_2 = \{e_1 = (1,0)^T, e_2 = (0,1)^T\}$ of $\mathbb{R}^2$.

Solution:

Using the definition of a matrix of a linear map w.r.t. bases of the domain and range we get

$$[f_6]_{K_2,K_2} = \big([f_6(e_1)]_{K_2} \quad [f_6(e_2)]_{K_2}\big) = \big([f_6((1,0)^T)]_{K_2} \quad [f_6((0,1)^T)]_{K_2}\big) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Alternatively, we could have used the theorem about computation of a linear map using $[f_6]_{K_2,K_2}$ and derive the same result from the definition of f_6.

Problem 4. Consider the basis $B_1 = \{(-1,0,3)^T, (2,-2,2)^T, (0,1,-3)^T\}$ of $\mathbb{R}^3$. Find the matrix of $f : \mathbb{R}^3 \to \mathbb{R}^3$ w.r.t. the basis B_1 (i.e., $[f]_{B_1,B_1}$) if you know that f maps the basis vectors as follows (note that all vectors are scaled by a factor of 2):

$$f((-1,0,3)^T) = (-2,0,6)^T,$$
$$f((2,-2,2)^T) = (4,-4,4)^T,$$
$$f((0,1,-3)^T) = (0,2,-6)^T.$$

For x with coordinates $[x]_{B_1} = (1, 2, -1)^T$, use the matrix $[f]_{B_1,B_1}$ to compute the coordinates $[f(x)]_{B_1}$ of the image of x under f w.r.t. B_1.

Solution:

We will construct the matrix $F = [f]_{B_1,B_1}$ using its definition. To compute the first column of F, we map the first basis vector $x_1 = (-1, 0, 3)^T$ to $f((-1, 0, 3)^T) = (-2, 0, 6)^T$ and compute the coordinates of the image w.r.t. basis B_1. Thus, we need to solve a linear system $Ax = b$ with the matrix

$$\left(\begin{array}{ccc|c} | & | & | & | \\ x_1 & x_2 & x_3 & f(x_1) \\ | & | & | & | \end{array}\right) \sim \left(\begin{array}{ccc|c} -1 & 2 & 0 & -2 \\ 0 & -2 & 1 & 0 \\ 3 & 2 & -3 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

where the basis vectors from B_1 form the columns of A and $f(x_1)$ is the right hand side b.

Note that we can compute all the vectors "in parallel" by manipulating the following block matrix

$$\left(\begin{array}{ccc|ccc} | & | & | & | & | & | \\ B_{U_1} & B_{U_2} & B_{U_3} & f(x_1) & f(x_2) & f(x_3) \\ | & | & | & | & | & | \end{array}\right) \sim \left(\begin{array}{ccc|ccc} | & | & | & | & | & | \\ x_1 & x_2 & x_3 & f(x_1) & f(x_2) & f(x_3) \\ | & | & | & | & | & | \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} -1 & 2 & 0 & -2 & 4 & 0 \\ 0 & -2 & 1 & 0 & -4 & 2 \\ 3 & 2 & -3 & 6 & 4 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right).$$

We can read off the result from the right block of the RREF, i.e.,

$$F = [f]_{B_1,B_1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

On a high level, it makes sense that the matrix is a multiple of the identity matrix, as the map simply takes the basis vectors in B_1 and scales them by a factor of two.

We can compute the coordinates of $f(x)$ w.r.t. B_1 using $[x]_{B_1} = (1, 2, -1)^T$ as $F[x]_{B_1} = [f]_{B_1,B_1}[x]_{B_1} = [f(x)]_{B_1} = (2, 4, -2)^T$.

Problem 5. For the linear map f from the previous problem, find the matrix $[f]_{B_1,B_2}$ of f w.r.t. the bases

$$B_1 = \{x_1 = (-1, 0, 3)^T, x_2 = (2, -2, 2)^T, x_3 = (0, 1, -3)^T\} \text{ and}$$
$$B_2 = \{y_1 = (-1, 1, 0)^T, y_2 = (0, 1, -1)^T, y_3 = (1, 0, 1)^T\}.$$

For x with coordinates $[x]_{B_1} = (1, 2, -1)^T$, use the matrix $[f]_{B_1,B_2}$ to compute the coordinates $[f(x)]_{B_2}$ of the image of x under f w.r.t. B_2.

Solution:
Again, we will construct the matrix $[f]_{B_1,B_2}$ using its definition. The computation is similar to the previous problem with the difference that we need to compute the coordinates of the images of basis vectors from B_1 w.r.t. basis B_2. Thus, we need to solve the following system

$$\left(\begin{array}{ccc|ccc} | & | & | & | & | & | \\ y_1 & y_2 & y_3 & f(x_1) & f(x_2) & f(x_3) \\ | & | & | & | & | & | \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} -1 & 0 & 1 & -2 & 4 & 0 \\ 1 & 1 & 0 & 0 & -4 & 2 \\ 0 & -1 & 1 & 6 & 4 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -2 & -2 \\ 0 & 1 & 0 & -4 & -2 & 4 \\ 0 & 0 & 1 & 2 & 2 & -2 \end{array}\right).$$

The matrix is $[f]_{B_1,B_2} = \begin{pmatrix} -2 & -2 & -2 \\ -4 & -2 & 4 \\ 2 & 2 & -2 \end{pmatrix}$.

Finally, for the given vector x with coordinates $[x]_{B_1} = (1, 2, -1)^T$, we compute

$$[f(x)]_{B_2} = [f]_{B_1,B_2}[x]_{B_1} = (2, -12, 8)^T.$$

Problem 6. For the bases B_1 and B_2 from the previous problem, find the change of basis matrix $[id]_{B_1,B_2}$ that transforms coordinates w.r.t. B_1 into coordinates w.r.t. B_2. For x with coordinates $[x]_{B_1} = (1, 2, -1)^T$, use the change of basis matrix $[id]_{B_1,B_2}$ to compute the coordinates $[x]_{B_2}$ of x w.r.t. B_2.

Solution:
We can proceed as above with the main difference that the transformation is the identical transformation. We compute the change of basis matrix as follows:

$$\left(\begin{array}{ccc|ccc} | & | & | & | & | & | \\ y_1 & y_2 & y_3 & x_1 & x_2 & x_3 \\ | & | & | & | & | & | \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} -1 & 0 & 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 & -2 & 1 \\ 0 & -1 & 1 & 3 & 2 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right).$$

The matrix is $[id]_{B_1,B_2} = \begin{pmatrix} 2 & -1 & -1 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix}$.

Finally, we transform the coordinates $[x]_{B_1} = (1, 2, -1)^T$ using the matrix $[id]_{B_1,B_2}$ and we get

$$[id]_{B_1,B_2}[x]_{B_1} = [id(x)]_{B_2} = [x]_{B_2} = (1, -6, 4)^T.$$

To check the result, we could also solve the corresponding system that computes the coordinates of x w.r.t. B_2 directly.

Problem 7. How about transforming the coordinates $[x]_{B_2}$ of x w.r.t. B_2 into coordinates w.r.t. B_1? Find the change of basis matrix $[id]_{B_2,B_1}$ that transforms coordinates w.r.t. B_2 into coordinates w.r.t. B_1.

For x with coordinates $[x]_{B_2} = (1, -6, 4)^T$, use the matrix $[id]_{B_2,B_1}$ to compute the coordinates $[x]_{B_1}$ of x w.r.t. B_1.

Solution:
We simply need to swap the blocks of the matrix constructed in the previous problem.

$$\left(\begin{array}{ccc|ccc} | & | & | & | & | & | \\ x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \\ | & | & | & | & | & | \end{array}\right) \sim \left(\begin{array}{ccc|ccc} -1 & 2 & 0 & -1 & 0 & 1 \\ 0 & -2 & 1 & 1 & 1 & 0 \\ 3 & 2 & -3 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 & 4 \end{array}\right).$$

The matrix is $[id]_{B_2,B_1} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix}$.

The coordinates $[x]_{B_1}$ are then computed as

$$[id]_{B_2,B_1}[x]_{B_2} = [id(x)]_{B_1} = [x]_{B_1} = (1, 2, -1)^T.$$

Problem 8. Consider $f\colon \mathbb{Z}_5^3 \to \mathbb{Z}_5^3$ defined by the matrix

$$[f]_{B,K_3} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 4 & 0 & 3 \end{pmatrix}$$

w.r.t. the standard basis K_3 of $\mathbb{Z}_5^3$ and the basis $B = \{(3,2,1)^T, (1,3,4)^T, (2,2,2)^T\}$ of $\mathbb{Z}_5^3$.

Compute the matrix $[f]_{K_3,K_3}$ of f w.r.t. to the standard basis K_3 of $\mathbb{Z}_5^3$.

Solution:
Since $f = f \circ id$, we can compute

$$[f]_{K_3,K_3} = [f]_{B,K_3}[id]_{K_3,B} = [f]_{B,K_3}([id]_{B,K_3})^{-1} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 4 & 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & 2 \\ 1 & 4 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix}.$$

Problem 9. Consider $g\colon \mathbb{Z}_7^2 \to \mathbb{Z}_7^3$ defined by the matrix

$$[g]_{K_2,K_3} = \begin{pmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 6 \end{pmatrix}$$

w.r.t. to the standard bases K_2 of $\mathbb{Z}_7^2$ and K_3 of $\mathbb{Z}_7^3$.

Compute the matrix $[g]_{B_2,B_3}$ of g w.r.t. the bases $B_2 = \{(1,4)^T, (3,1)^T\}$ of $\mathbb{Z}_7^2$ and $B_3 = \{(1,1,2)^T, (1,0,3)^T, (6,0,5)^T\}$ of $\mathbb{Z}_7^3$.

Solution:
Note that $g = id \circ g \circ id$ and we can compute

$$[g]_{B_2,B_3} = [id]_{K_3,B_3}[g]_{K_2,K_3}[id]_{B_2,K_2} = ([id]_{B_3,K_3})^{-1}[g]_{K_2,K_3}[id]_{B_2,K_2},$$

where we can easily construct the last two change of basis matrices

$$[id]_{B_2,K_2} = \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix}, [id]_{B_3,K_3} = \begin{pmatrix} 1 & 1 & 6 \\ 1 & 0 & 0 \\ 2 & 3 & 5 \end{pmatrix}.$$

To complete the computation, we need to only compute the corresponding inverse and multiply the matrices:

$$[g]_{B_2,B_3} = \begin{pmatrix} 1 & 1 & 6 \\ 1 & 0 & 0 \\ 2 & 3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 0 & 0 \\ 5 & 6 \end{pmatrix}.$$

Problem 10. Consider $h\colon \mathbb{Z}_5^2 \to \mathbb{Z}_5^3$ defined by the matrix

$$[h]_{B_2,B_3} = \begin{pmatrix} 4 & 3 \\ 2 & 4 \\ 3 & 1 \end{pmatrix}$$

w.r.t. the bases $B_2 = \{(4,3)^T, (1,4)^T\}$ of $\mathbb{Z}_5^2$ and $B_3 = \{(1,1,1)^T, (1,4,0)^T, (4,0,1)^T\}$ of $\mathbb{Z}_5^3$.

Compute the matrix $[h]_{K_2,K_3}$ of h w.r.t. the standard bases K_2 of $\mathbb{Z}_5^2$ and K_3 of $\mathbb{Z}_5^3$.

Solution:
Similarly to above, note that $h = id \circ h \circ id$ and we can compute

$$\begin{aligned}[h]_{K_2,K_3} &= [id]_{B_3,K_3}[h]_{B_2,B_3}[id]_{K_2,B_2} = [id]_{B_3,K_3}[h]_{B_2,B_3}([id]_{B_2,K_2})^{-1} \\ &= \begin{pmatrix} 1 & 1 & 4 \\ 1 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & 3 \end{pmatrix}.\end{aligned}$$

Problem 11. For the linear maps f and h defined above, compute the matrix $[f \circ h]_{K_2,K_3}$ of the composed map $f \circ h\colon \mathbb{Z}_5^2 \to \mathbb{Z}_5^3$ w.r.t. the standard bases K_2 of $\mathbb{Z}_5^2$ and K_3 of $\mathbb{Z}_5^3$.

Solution:

Since we know both matrices, we can compute the required matrix as their product:

$$[f \circ h]_{K_2,K_3} = [f]_{K_3,K_3}[h]_{K_2,K_3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \\ 4 & 1 \end{pmatrix}.$$