NMAI057 – Linear algebra 1

Tutorial 9

Row space, column space, and kernel

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Problem 1. Compute the dimension and find the basis for the row space $\mathcal{R}(A)$, the column space $\mathcal{C}(A)$, and the kernel $\mathrm{Ker}(A)$ of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}.$$

Problem 2. Over $\mathbb{R}$, $\mathbb{Z}_5$, and $\mathbb{Z}_7$, decide and justify whether for $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ it holds that

(a) $(1,2)^T \in \mathrm{Ker}(A)$,

(b) $(1,2)^T \in \mathcal{C}(A)$.

Problem 3. Construct a matrix A such that:

(a) $\mathcal{R}(A)$ contains vectors $(1,1)^T, (1,2)^T$ and $\mathcal{C}(A)$ contains $(1,0,0)^T$, $(0,0,1)^T$.

(b) The basis of both $\mathcal{R}(A)$ and $\mathcal{C}(A)$ is $(1,1,1)^T$ and the basis of $\mathrm{Ker}(A)$ is $(1,-2,1)^T$.

Problem 4. Decide and justify whether for all $A, B \in \mathbb{R}^{n\times n}$ it holds that

(a) $\mathcal{C}(A) = \mathcal{C}(B)$ implies $\mathrm{RREF}(A) = \mathrm{RREF}(B)$,

(b) $\mathrm{RREF}(A) = \mathrm{RREF}(B)$ implies $\mathcal{C}(A) = \mathcal{C}(B)$.

Problem 5. Choose a basis B of $V = \mathrm{span}\{v_1, v_2, v_3, v_4\}$ from vectors

$$v_1 = (3,1,5,4)^T,\ v_2 = (2,2,3,3)^T,\ v_3 = (1,-1,2,1)^T,\ v_4 = (1,3,1,1)^T.$$

For the vectors not in your basis B, compute their coordinates w.r.t. B.

Problem 6. Decide and justify whether for all $A, B \in \mathbb{R}^{m\times n}$ it holds that $\mathrm{rank}(A+B) \le \mathrm{rank}(A) + \mathrm{rank}(B)$.
(Hint: What is the relationship between $\mathcal{C}(A) + \mathcal{C}(B)$ and $\mathcal{C}(A+B)$?)

Problem 7. In terms of inclusion, what is the relationship between $\mathrm{Ker}(AB)$ and $\mathrm{Ker}(B)$ for matrices

(a) $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$,

(b) $A \in \mathbb{R}^{n\times n}$ regular and $B \in \mathbb{R}^{n\times p}$?