NMAI057 – Linear algebra 1

Tutorial 7 & 8

Subspaces and linear independence

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Problem 1. Decide and **justify** for what parameters $a \in \mathbb{Z}_7$ is the set

$$S_a = \{(x, y, z)^T : x + 2y - 3z = a\}$$

a subspace of the vector space $\mathbb{Z}_7^3$.

What is the cardinality of this vector space?

Problem 2. Over $\mathbb{Z}_{11}$, find the intersection of the subspaces of $\mathbb{Z}_{11}^4$ defined as 1) the solution set of the system $Ax = 0$ and 2) the span of the set of vectors $\{v_1, v_2, v_3\}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 3 & 5 & 2 & 1 \end{pmatrix}, \; v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 1 \\ 0 \\ 9 \\ 0 \end{pmatrix}.$$

Problem 3. Decide and **justify** whether the set of all univariate polynomials with coefficients in $\mathbb{Z}_3$ and degree lesser or equal to 10 is a vector space (w.r.t. the natural operations of addition of vectors and multiplication by scalar).

What is the cardinality of the set?

Problem 4. Decide and **justify** whether the following vectors are linearly independent in $\mathbb{R}^3$:

(a) $(2, 3, -5)^T, (1, -1, 1)^T, (3, 2, -2)^T$.

(b) $(2, 0, 3)^T, (1, -1, 1)^T, (0, 2, 1)^T$.

Problem 5. Let u, v, w be linearly independent vectors in a vector space V over $\mathbb{R}$. Decide and **justify** whether the following sets of vectors are linearly independent

(a) $\{u, u + v, u + w\}$,

(b) $\{u - v, u - w, v - w\}$.

Problem 6. Let V be a vector space over a field $\mathbb{F}$ and $X \subseteq Y \subseteq V$. Decide and **justify** whether the following statements are true:

(a) If X is linearly independent then Y is linearly dependent.

(b) If X is linearly independent then Y is linearly independent.

(c) If X is linearly dependent then Y is linearly dependent.

(d) If Y is linearly independent then X is linearly independent.

(e) If Y is linearly dependent then X is linearly dependent.

Problem 7. Decide and **justify** whether $\{(0,1,1,1)^T, (1,0,1,1)^T, (1,1,0,1)^T, (1,1,1,0)^T\}$ is linearly independent in $\mathbb{R}^4$, respectively in $\mathbb{Z}_3^4$.

Problem 8. Let U, V be subspaces of a vector space W over $\mathbb{F}$. Prove that $U \cap V = \{o\}$ if and only if for all $x \in U + V$ there exists a unique choice of $u \in U, v \in V$ such that $x = u + v$.

Problem 9. Decide and **justify** whether the following sets of vectors are linearly independent in the vector space of univariate real functions $\mathbb{R} \to \mathbb{R}$ (over $\mathbb{R}$)

(a) $\{2x-1, x-2, 3x\}$,

(b) $\{x^2+2x+3, x+1, x-1\}$,

(c) $\{\sin x, \cos x\}$,

(d) $\{\sin(x+1), \sin(x+2), \sin(x+3)\}$,

(e) $\{\ln(x), \log_{10}(x), \log_2(x^2)\}$.