NMAI057 – Linear algebra 1

Tutorial 7 & 8

Subspaces and linear independence

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Problem 1. Decide and **justify** for what parameters $a \in \mathbb{Z}_7$ is the set

$$S_a = \{(x, y, z)^T : x + 2y - 3z = a\}$$

a subspace of the vector space $\mathbb{Z}_7^3$.

What is the cardinality of this vector space?

Solution:
For S_a to be a subspace of $\mathbb{Z}_7^3$, it must contain the zero vector $(0, 0, 0)^T$. Thus, it must hold that $a = 0 + 2 \cdot 0 - 3 \cdot 0 = 0$. We show that for $a = 0$ it is a subspace. Note that it remains to decide whether the set S_a is closed under addition of vectors and multiplication by a scalar from $\mathbb{Z}_7$.

Multiplication by scalar: For all $(x, y, z) \in S_0$ and $\alpha \in \mathbb{Z}_7$, it holds that $\alpha x + 2\alpha y - 3\alpha z = \alpha(x + 2y - 3z) = \alpha \cdot 0 = 0$. Thus, $\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z) \in S_0$.

Addition of vectors: For all $(x, y, z) \in S_0$ and $(x', y', z') \in S_0$, it follows by distributivity, commutativity and associativity of addition over $\mathbb{Z}_7$ that $(x + x') + 2(y + y') - 3(z + z') = (x + 2y - 3z) + (x' + 2y' - 3z) = 0 + 0 = 0$. Thus, $(x + x', y + y', z + z') \in S_0$.

Finally, we compute the cardinality of S_0. For any choice of x and y, we get a z (i.e., $z = \frac{x+2y}{3} = 5x + 3y$) satisfying $x + 2y - 3z = 0$. There are 7 distinct choices of x and 7 distinct choices of y and, therefore, there are $7 \cdot 7 = 49$ elements of S_0.

To summarize, S_a is a subspace only for $a = 0$ and in that case it has 49 elements.

Problem 2. Over $\mathbb{Z}_{11}$, find the intersection of the subspaces of $\mathbb{Z}_{11}^4$ defined as 1) the solution set of the system $Ax = 0$ and 2) the span of the set of vectors $\{v_1, v_2, v_3\}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 3 & 5 & 2 & 1 \end{pmatrix}, \ v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \ v_3 = \begin{pmatrix} 1 \\ 0 \\ 9 \\ 0 \end{pmatrix}.$$

Solution:
First, we solve the system

$$\begin{pmatrix}1&2&3&2\\3&5&2&1\end{pmatrix}\sim\begin{pmatrix}1&2&3&3\\0&-1&-7&-5\end{pmatrix}\sim\begin{pmatrix}1&0&0&-1\\0&-1&-7&-5\end{pmatrix}\sim\begin{pmatrix}1&0&0&10\\0&1&7&5\end{pmatrix}.$$

The solution set is $\left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}+r\cdot\begin{pmatrix}1\\6\\0\\1\end{pmatrix}+s\cdot\begin{pmatrix}0\\4\\1\\0\end{pmatrix} : r,s\in\mathbb{Z}_{11}\right\}$.

Our task is to find out which of the vectors in the solution set can be expressed as $a_1v_1+a_2v_2+a_3v_3$, where $a_1,a_2,a_3\in\mathbb{Z}_{11}$. Let's denote $w_1=\begin{pmatrix}1\\6\\0\\1\end{pmatrix}$ and $w_2=\begin{pmatrix}0\\4\\1\\0\end{pmatrix}$.

In other words, we need to solve $a_1v_1+a_2v_2+a_3v_3=rw_1+sw_2$. Equivalently, $a_1v_1+a_2v_2+a_3v_3+r(-w_1)+s(-w_2)=0$:

$$\begin{pmatrix}1&0&1&-1&0\\2&2&0&-6&-4\\1&3&-2&0&-1\\1&1&0&-1&0\end{pmatrix}\sim\begin{pmatrix}1&0&1&-1&0\\0&2&-2&-4&-4\\0&3&-3&1&-1\\0&1&-1&0&0\end{pmatrix}\sim\begin{pmatrix}1&0&1&-1&0\\0&1&-1&-2&-2\\0&3&-3&1&-1\\0&1&-1&0&0\end{pmatrix}$$
$$\sim\begin{pmatrix}1&0&1&-1&0\\0&1&-1&-2&-2\\0&0&0&7&6\\0&0&0&2&2\end{pmatrix}\sim\begin{pmatrix}1&0&1&-1&0\\0&1&-1&-2&-2\\0&0&0&1&4\\0&0&0&2&2\end{pmatrix}$$
$$\sim\begin{pmatrix}1&0&1&-1&0\\0&1&-1&-2&-2\\0&0&0&1&4\\0&0&0&0&-6\end{pmatrix}=\begin{pmatrix}1&0&1&10&0\\0&1&10&9&9\\0&0&0&1&4\\0&0&0&0&5\end{pmatrix}.$$

It follows that $r=0$ and $s=0$. Thus, the only vector in the intersection is $0\cdot w_1+0\cdot w_2$, i.e., the zero vector.

Problem 3. Decide and **justify** whether the set of all univariate polynomials with coefficients in $\mathbb{Z}_3$ and degree lesser or equal to 10 is a vector space (w.r.t. the natural operations of addition of vectors and multiplication by scalar).

What is the cardinality of the set?

Solution:
True. Follows by a straightforward verification using the characterization of a vector space as a set which contains the zero vector and is closed under addition of vectors and multiplication by a scalar.

The cardinality of the set is 3^{11} (there are three possibilities for each coefficient for the eleven monomials $x^{10},\dots,x^0=1$).

Problem 4. Decide and **justify** whether the following vectors are linearly independent in $\mathbb{R}^3$:

(a) $(2,3,-5)^T,(1,-1,1)^T,(3,2,-2)^T$.

(b) $(2,0,3)^T,(1,-1,1)^T,(0,2,1)^T$.

Solution:

(a) We are looking for coefficients $a, b, c \in \mathbb{R}$ such that

$$a \cdot \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} + b \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c \cdot \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, the problem of verifying whether the given vectors are linearly independent is equivalent to solving the following homogeneous system of linear equations

$$\left(\begin{array}{rrr|r} 2 & 1 & 3 & 0 \\ 3 & -1 & 2 & 0 \\ -5 & 1 & -2 & 0 \end{array}\right).$$

Note that the given vectors form the columns of the matrix. By solving the system, we verify that there is only the trivial solution $a = b = c = 0$ and the vectors are linearly independent.

(b) Similarly to (a), we form the corresponding system

$$\left(\begin{array}{rrr|r} 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 3 & 1 & 1 & 0 \end{array}\right).$$

The Gaussian elimination process gives

$$\left(\begin{array}{rrr|r} 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus, the system has also some non-trivial solutions and the vectors are linearly dependent. To exemplify the nontrivial solution, the parametric description of the solution set is $(-t, 2t, t)^T$ for a real parameter $t \in \mathbb{R}$. For $a = -1, b = 2$, and $c = 1$, we get

$$-1 \cdot \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 5. Let u, v, w be linearly independent vectors in a vector space V over $\mathbb{R}$. Decide and **justify** whether the following sets of vectors are linearly independent

(a) $\{u, u + v, u + w\}$,

(b) $\{u - v, u - w, v - w\}$.

Solution:

(a) Similarly to the previous problem, we are looking for coefficients $a, b, c \in \mathbb{R}$ such that

$$o = au + b(u+v) + c(u+w) = (a+b+c)u + bv + cw.$$

Since u, v, w linearly independent, it must hold that $a+b+c=0$, $b=0$, and $c=0$. Thus, also $a=0$. We can conclude that the set $\{u, u+v, u+w\}$ is linearly independent.

(b) Analogously, we are looking for coefficients $a, b, c \in \mathbb{R}$ such that

$$o = a(u-v) + b(u-w) + c(v-w) = (a+b)u + (-a+c)v + (-b-c)w.$$

Thus, $a+b=0$, $-a+c=0$, and $-b-c=0$. By solving the corresponding system, we get a parametric description of the solution set as $(t, -t, t)^T$ for a real parameter $t \in \mathbb{R}$. Ve can conclude that the set $\{u-v, u-w, v-w\}$ is linearly dependent, and a non-trivial combination which is equal to the zero vector using can be obtained using for example the coefficients $(1, -1, 1)^T$.

Problem 6. Let V be a vector space over a field $\mathbb{F}$ and $X \subseteq Y \subseteq V$. Decide and **justify** whether the following statements are true:

(a) If X is linearly independent then Y is linearly dependent.

(b) If X is linearly independent then Y is linearly independent.

(c) If X is linearly dependent then Y is linearly dependent.

(d) If Y is linearly independent then X is linearly independent.

(e) If Y is linearly dependent then X is linearly dependent.

<u>Solution:</u>

The general "rule" is that linear independence is preserved "downwards" and linear dependence is preserved "upwards" w.r.t. inclusion.

(a) False: $X = \{(1,0)^T\}$ and $Y = \{(1,0)^T, (0,1)^T\}$ are both linearly independent in $\mathbb{R}^2$.

(b) False: $X = \{(1,0)^T\}$ is linearly independent but $Y = \{(1,0)^T, (2,0)^T\}$ is linearly dependent in $\mathbb{R}^2$.

(c) True. Let $X = \{v_1, \ldots, v_\ell\}$ and $Y = \{v_1, \ldots, v_\ell, w_1 \ldots, w_k\}$ be two subsets of V. By the claim, X is linearly dependent and, thus, there exist $\alpha_1, \ldots, \alpha_\ell \in \mathbb{F}$ such that $(\alpha_1, \ldots, \alpha_\ell)^T \neq (0, \ldots, 0)^T$ and

$$\sum_{i \in [\ell]} \alpha_i x_i = o.$$

Let $\beta_{\ell+1}, \ldots, \beta_k = (0, \ldots, 0)$. It also holds that $(\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_k) \neq (0, \ldots, 0)$ and

$$\sum_{i \in [\ell]} \alpha_i v_i + \sum_{j \in [k]} \beta_j w_j = o$$

is a non-trivial combination of vectors in Y equal to the zero vector o. Therefore, the set Y is also linearly dependent.

(d) True. A variation of (c).

(e) False: $Y = \{(1,0)^T, (2,0)^T\}$ is linearly dependent but $X = \{(1,0)^T\}$ is linearly independent in $\mathbb{R}^2$.

Problem 7. Decide and **justify** whether $\{(0,1,1,1)^T, (1,0,1,1)^T, (1,1,0,1)^T, (1,1,1,0)^T\}$ is linearly independent in $\mathbb{R}^4$, respectively in $\mathbb{Z}_3^4$.

Solution:
The problem can be solved similarly to Problem 4 while taking into account the underlying field ($\mathbb{R}$ or $\mathbb{Z}_3$). The vectors are linearly independent in $\mathbb{R}^4$ and they are linearly dependent on $\mathbb{Z}_3^4$. Thus, linear independence is not invariant w.r.t. the underlying field.

Problem 8. Let U, V be subspaces of a vector space W over $\mathbb{F}$. Prove that $U \cap V = \{o\}$ if and only if for all $x \in U + V$ there exists a unique choice of $u \in U, v \in V$ such that $x = u + v$.

Solution:
Suppose that there are two distinct ways to express x as

$$u_1 + v_1 = x = u_2 + v_2,$$

for some $u_1, u_2 \in U$ and $v_1, v_2 \in V$. The equality gives

$$u_1 - u_2 = v_2 - v_1.$$

Note that the vector $u_1 - u_2$ lies in the subspace U and the vector $v_2 - v_1$ lies in the subspace V.

Then $v + v = x = 2v + o$ are two distinct ways of expressing a vector $x \in U + V$, a contradiction of the assumption of the statement. If $U \cap V\{o\}$ then $u_1 - u_2 = v_2 - v_1 = o$. And we get that $u_1 = u_2$ and $v_1 = v_2$, a contradiction to $u_1 + v_1$ and $u_2 + v_2$ being two *distinct* ways to express x.

To prove the other implication, suppose there exists a non-zero vector $v \in U \cap V$. Then $v + v = x = 2v + o$ are two distinct ways of expressing a vector $x \in U + V$, a contradiction of the assumption of the statement.

Problem 9. Decide and **justify** whether the following sets of vectors are linearly independent in the vector space of univariate real functions $\mathbb{R} \to \mathbb{R}$ (over $\mathbb{R}$)

(a) $\{2x - 1, x - 2, 3x\}$,

(b) $\{x^2 + 2x + 3, x + 1, x - 1\}$,

(c) $\{\sin x, \cos x\}$,

(d) $\{\sin(x + 1), \sin(x + 2), \sin(x + 3)\}$,

(e) $\{\ln(x), \log_{10}(x), \log_2(x^2)\}$.

Solution:

(a) Denote $f(x) = 2x - 1, g(x) = x - 2$, and $h(x) = 3x$. We are looking for $a, b, c \in \mathbb{R}$ such that $a \cdot f(x) + b \cdot g(x) + c \cdot h(x) = 0$ for all $x \in \mathbb{R}$, i.e.,

$$a \cdot (2x - 1) + b \cdot (x - 2) + c \cdot 3x = (2a + b + 3c) \cdot x + (-a - 2b) = 0.$$

The equality holds for all $x \in \mathbb{R}$ if and only if

$$\begin{aligned} 2a + b + 3c &= 0 \\ -a - 2b &= 0. \end{aligned}$$

A non-trivial solution of the system is for example $(-2, 1, 1)^T$. Thus, the functions are linearly dependent.

(b) Similarly, we are looking for $a, b, c \in \mathbb{R}$ such that

$$a \cdot (x^2 + 2x + 3) + b \cdot (x + 1) + c \cdot (x - 1) = a \cdot x^2 + (2a + b + c) \cdot x + (b - c) = 0.$$

This corresponds to the homogeneous system

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right),$$

which has only the trivial solution $(0, 0, 0)$. Therefore, the functions are linearly independent.

(c) We try to satisfy the equation $a \sin x + b \cos x = 0$. Note that for $x = 0$, we get $b = 0$ since $\sin 0 = 0$ and $\cos 0 = 1$. For $x = \frac{\pi}{2}$, we get $a = 0$ since $\sin \frac{\pi}{2}$ and $\cos \frac{\pi}{2} = 1$. Thus, the only coefficients which make the functions to sum up to the identically zero function (i.e., for all $x \in \mathbb{R}$) are $a = 0$ and $b = 0$ and the functions are linearly independent.

(d) Standard identities for $\sin x$ give:

$$\begin{aligned} \sin(x + 1) &= \sin(x) \cdot \cos(1) + \cos(x) \cdot \sin(1), \\ \sin(x + 2) &= \sin(x) \cdot \cos(2) + \cos(x) \cdot \sin(2), \\ \sin(x + 3) &= \sin(x) \cdot \cos(3) + \cos(x) \cdot \sin(3). \end{aligned}$$

When checking the linear independence of the functions, we get the equation:

$$\begin{aligned} 0 &= a \cdot \sin(x + 1) + b \cdot \sin(x + 2) + c \cdot \sin(x + 3) \\ &= \big(a \cdot \cos(1) + b \cdot \cos(2) + c \cdot \cos(3)\big) \cdot \sin(x) \\ &\quad + \big(a \cdot \sin(1) + b \cdot \sin(2) + c \cdot \sin(3)\big) \cdot \cos(x). \end{aligned}$$

Since sin and cos are linearly independent, it must hold that

$$\begin{aligned} a \cos(1) + b \cos(2) + c \cos(3) &= 0, \\ a \sin(1) + b \sin(2) + c \sin(3) &= 0. \end{aligned}$$

We get a homogeneous system with two equations in three unknowns which must have a non-trivial solution. Thus, the functions are linearly dependent.

(e) We can use the following identities $\log_{10}(2x) = \frac{\ln x + \ln 2}{\ln 10}$ and $\log_2(x^2) = \frac{2 \ln x}{\ln 2}$. Thus, in the equation $a \cdot \ln(x) + b \cdot \log_{10}(2x) + c \cdot \log_2(x^2) = 0$, the first term and the last term are multiples of each other. For example $(a, b, c) = (-2, 0, \ln 2)$ is a non-trivial solution of the equation. Thus, the functions are linearly dependent.