NMAI057 – Linear algebra 1

Tutorial 5 & 6

Groups and Fields

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Problem 1. Decide and justify, whether the following are groups:

(a) $(\mathbb{Q}, \cdot)$,

(b) $(\mathbb{Q}, -)$,

(c) $(\mathbb{Q} \setminus \{0\}, \circ)$, where for all $a, b \in \mathbb{Q}$, $a \circ b = |ab|$,

(d) $(\mathbb{Q}, \circ)$, where for all $a, b \in \mathbb{Q}$, $a \circ b = \frac{a+b}{2}$,

(e) $(\mathbb{Q}, \circ)$, where for all $a, b \in \mathbb{Q}$, $a \circ b = a + b + 3$,

(f) $(\mathcal{F}, +)$, i.e., the set of all real functions with one variable $\mathcal{F}$ together with the operation of addition of functions,

(g) the set of all rotations around the origin in $\mathbb{R}^2$ together with the operation of function composition,

(h) the set of all translations (shifts) in $\mathbb{R}^2$ together with the operation of function composition.

Problem 2. Fill the table for binary operation $\circ$ on set $\mathbb{G}$ so that $(\mathbb{G}, \circ)$ is a group with neutral element 0. Justify.

(a)

$\circ$	0	1
0		
1		

(b)

$\circ$	0	1	2
0			
1			
2			

(c)

$\circ$	0
0	

(d)

$\circ$	0	1	2	3
0				
1		0		
2				
3				

Problem 3. Let $(\mathbb{G}, \circ)$ be a group and $x \in \mathbb{G}$. Decide and justify whether $(\mathbb{G}, *)$ is a group with the binary operation $*$ defined for all $a, b \in \mathbb{G}$ as $a * b = a \circ x \circ b$.

Problem 4. Decide and justify whether the following are Abelian (commutative) groups:

(a) The set $\{\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) \mid z \in \mathbb{Z}\}$ together with matrix product.

(b) The set $\{\left(\begin{smallmatrix} a & a \\ a & a \end{smallmatrix}\right) \mid a \in \mathbb{R} \setminus \{0\}\}$ together with matrix product.

Problem 5. Simplify the following expressions:

(a) $((2^{-1}+1)4)^{-1}, 4/3$ over $\mathbb{Z}_5$,

(b) $6+7, -7, 6 \cdot 7, 7^{-1}, 6/7$ over $\mathbb{Z}_{11}$.

Problem 6. Over $\mathbb{Z}_5$, find the set of all solutions of the system

$$\begin{aligned} 3x + 2y + z &= 1 \\ 4x + y + 3z &= 3 \end{aligned}$$

and compute its cardinality.

Problem 7. Find the multiplicative inverses 9^{-1} and 12^{-1} in $\mathbb{Z}_{31}$.

Problem 8. Over $\mathbb{Z}_7$, compute the matrix power A^{100} for $A = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 4 \end{smallmatrix}\right)$.