NMAI057 – Linear algebra 1

Tutorial 5 & 6

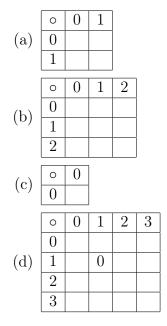
Groups and Fields

Date: November 9 and 16, 2021

TA: Pavel Hubáček

Problem 1. Decide and justify, whether the following are groups:

- (a) $(\mathbb{Q}, \cdot),$
- (b) $(\mathbb{Q}, -),$
- (c) $(\mathbb{Q} \setminus \{0\}, \circ)$, where for all $a, b \in \mathbb{Q}$, $a \circ b = |ab|$,
- (d) (\mathbb{Q}, \circ) , where for all $a, b \in \mathbb{Q}$, $a \circ b = \frac{a+b}{2}$,
- (e) (\mathbb{Q}, \circ) , where for all $a, b \in \mathbb{Q}$, $a \circ b = a + b + 3$,
- (f) $(\mathcal{F}, +)$, i.e., the set of all real functions with one variable \mathcal{F} together with the operation of addition of functions,
- (g) the set of all rotations around the origin in \mathbb{R}^2 together with the operation of function composition,
- (h) the set of all translations (shifts) in \mathbb{R}^2 together with the operation of function composition.
- **Problem 2.** Fill the table for binary operation \circ on set \mathbb{G} so that (\mathbb{G}, \circ) is a group with neutral element 0. Justify.



Problem 3. Let (\mathbb{G}, \circ) be a group and $x \in \mathbb{G}$. Decide and justify whether $(\mathbb{G}, *)$ is a group with the binary operation * defined for all $a, b \in \mathbb{G}$ as $a * b = a \circ x \circ b$.

Problem 4. Decide and justify whether the following are Abelian (commutative) groups:

- (a) The set $\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{Z}\}$ together with matrix product.
- (b) The set $\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \mid a \in \mathbb{R} \setminus \{0\} \}$ together with matrix product.

Problem 5. Simplify the following expressions:

(a) $((2^{-1}+1)4)^{-1}, 4/3 \text{ over } \mathbb{Z}_5,$ (b) $6+7, -7, 6\cdot 7, 7^{-1}, 6/7 \text{ over } \mathbb{Z}_{11}.$

Problem 6. Over \mathbb{Z}_5 , find the set of all solutions of the system

$$3x + 2y + z = 1$$
$$4x + y + 3z = 3$$

and compute its cardinality.

Problem 7. Find the multiplicative inverses 9^{-1} and 12^{-1} in \mathbb{Z}_{31} .

Problem 8. Over \mathbb{Z}_7 , compute the matrix power A^{100} for $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$.