NMAI057 – Linear algebra 1

Tutorial 3

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Example 1: Compute the following expressions:

(a) $2A$
(b) $A + B$
(c) $A - B$
(d) C^T
(e) Cv
(f) AB
(g) BC

for

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix},\ B = \begin{pmatrix} -1 & -1 \\ 0 & 3 \end{pmatrix},\ C = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix},\ v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Solution:

(a) When the matrix A is of order $n \times m$, the resulting matrix will also be of order $n \times m$. By definition, we get the resulting matrix by multiplying each element of A by the constant 2. Thus,

$$2A = 2\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 & 2 \cdot 2 \\ 2 \cdot 2 & 2 \cdot -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & -2 \end{pmatrix}.$$

(b) Sum of two matrices A, B is defined only for matrices of the same dimensions (note that both are of the same order 2×2). The resulting matrix is of the same order as A (respectively B), i.e., 2×2; and, by definition, it is obtained via component-wise addition. Thus,

$$A + B = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 + (-1) & 2 + (-1) \\ 2 + 0 & -1 + 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}.$$

(c) Similarly to the previous case, the matrices A, B must be of the same order and the resulting matrix is also of the same order. The resulting matrix is obtained via component-wise subtraction of the matrices A and B. Thus,

$$A - B = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} -1 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 - (-1) & 2 - (-1) \\ 2 - 0 & -1 - 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & -4 \end{pmatrix}.$$

Note that we can interpret subtracting matrices as adding the matrices A and $(-1)B$.

(d) If the original matrix is of the order $n \times m$ then the transpose is of order $m \times n$. The element at position i, j in the transpose is equal to the element at position j, i in the original matrix. Thus,

$$C^T = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix}^T = \begin{pmatrix} 3 & 2 \\ 0 & -2 \\ 1 & 0 \end{pmatrix} .$$

(e) If the matrix C is of order $m \times n$ then the vector v must be of dimension n and their product is a vector of dimension m. We are given a matrix C of order 2×3 and a vector v of dimension 3, and, therefore, the dimensions of C and v allow to compute the product which is vector of dimension 2. Thus,

$$Cv = \begin{pmatrix} 3 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 \\ 2 \cdot 1 + (-2) \cdot 2 + 0 \cdot 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix} .$$

Note that product of a matrix and a vector is a special case of a product of matrices. In the above case, of the matrix C of order 2×3 and a matrix of order 3×1 corresponding to the vector v.

(f) If the matrix A is of order $m \times n$ then the matrix B must be of order $n \times o$ and their product is a matrix of order $m \times o$. We are given matrices A and B of order 2×2 (their product is defined) and their product is of order 2×2.

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot (-1) + 2 \cdot 0 & 1 \cdot (-1) + 2 \cdot 3 \\ 2 \cdot (-1) + (-1) \cdot 0 & 2 \cdot (-1) + (-1) \cdot 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 5 \\ -2 & -5 \end{pmatrix} . \end{aligned}$$

(g) If the matrix B is of order $m \times n$ then the matrix C must be of order $n \times o$ and their product is a matrix of order $m \times o$. The matrix B is of order 2×2 and the matrix C of order 2×3 (their product is defined) and their product is of order 2×3.

$$\begin{aligned} BC &= \begin{pmatrix} -1 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (-1) \cdot 3 + (-1) \cdot 2 & (-1) \cdot 0 + (-1) \cdot (-2) & (-1) \cdot 1 + (-1) \cdot 0 \\ 0 \cdot 3 + 3 \cdot 2 & 0 \cdot 0 + 3 \cdot (-2) & 0 \cdot 1 + 3 \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 2 & -1 \\ 6 & -6 & 0 \end{pmatrix} . \end{aligned}$$

Example 2: Prove or disprove the following:

(a) For all matrices $A \in \mathbb{R}^{m \times n}$, $A + A = 2A$.

(b) For all square matrices $A \in \mathbb{R}^{m \times m}$, $A = A^T$.

Solution:

(a) First, we verify that both sides of the identity are well defined.

For the left side, we need to verify that the addition is well defined, i.e., the matrices must be of the same order. As we are adding the matrix A to itself, and the dimensions are trivially identical. Thus, the left side of the identity is well defined for all A.

On the right side we are multiplying A with a constant 2. This operation can be performed with an arbitrary matrix A, and, thus, the right side of the identity is well defined for all matrices A.

It is important to not forget this step! It might be the case that the identity holds if and only if both sides are well defined. Consider for example the statement: for all $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{o\times p}$: $A + B - B = A$

Second, we verify the identity by verifying that 1) the two sides of the identity are of the same dimension and 2) they are component-wise equal. As for the left side, the result of adding two matrices of order $m \times n$ is, by definition, a matrix of order $m \times n$. As for the right side, the result of multiplying a matrix of order $m \times n$ by a constant is, by definition, a matrix of order $m \times n$. Therefore, both sides are of the same order.

Fianlly, we verify that the left and right side are component-wise equal. For all row indices i and column indices j, show that:

$$\begin{aligned}
[A + A]_{i,j} &= A_{i,j} + A_{i,j} && \text{(by definition of matrix addition)}\\
&= 2A_{i,j} && \text{(adding two real values)}\\
&= [2A]_{i,j}\ . && \text{(by definition of multiplication of matrix by constant)}
\end{aligned}$$

(b) To dispove the statement, we give a counterexample. We present a matrix A that satisfies the assumptions but violates the statement.

For example, we can choose $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The only assumption is that the matrix is square, which is satisfied by our choice of A. Moreover, $A^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A$.

If it is not clear that our counter-example satisfies some of the assumptions of the statement then we need to prove it satisfies the assumptions.

It is interesting to find also the minimal additional assumptions which make the statement true. In this case, we would need A to be symmetric.

Problem 1. Compute $(-1)A + 2BC$ for matrices

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}, B = \begin{pmatrix} 5 & 9 \\ 2 & 7 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}.$$

Solution:

$$
\begin{aligned}
(-1)\begin{pmatrix}3 & 1\\ 4 & 1\end{pmatrix} + 2\begin{pmatrix}5 & 9\\ 2 & 7\end{pmatrix}\begin{pmatrix}1 & -1\\ 2 & 2\end{pmatrix}
&= \begin{pmatrix}(-1)\cdot 3 & (-1)\cdot 1\\ (-1)\cdot 4 & (-1)\cdot 1\end{pmatrix} + 2\begin{pmatrix}5\cdot 1+9\cdot 2 & 5\cdot(-1)+9\cdot 2\\ 2\cdot 1+7\cdot 2 & 2\cdot(-1)+7\cdot 2\end{pmatrix}\\
&= \begin{pmatrix}-3 & -1\\ -4 & -1\end{pmatrix} + 2\begin{pmatrix}23 & 13\\ 16 & 12\end{pmatrix}\\
&= \begin{pmatrix}-3 & -1\\ -4 & -1\end{pmatrix} + \begin{pmatrix}2\cdot 23 & 2\cdot 13\\ 2\cdot 16 & 2\cdot 12\end{pmatrix}\\
&= \begin{pmatrix}-3 & -1\\ -4 & -1\end{pmatrix} + \begin{pmatrix}46 & 26\\ 32 & 24\end{pmatrix}\\
&= \begin{pmatrix}-3+46 & -1+26\\ -4+32 & -1+24\end{pmatrix} = \begin{pmatrix}43 & 25\\ 28 & 23\end{pmatrix}
\end{aligned}
$$

Problem 2. Solve the systems of linear equations $(A \mid b)$ and $(B \mid c)$ given by

$A = \begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}$ a $b = \begin{pmatrix}2\\ 1\end{pmatrix}$, and

$B = \begin{pmatrix}1 & 1 & 2\\ 1 & 1 & 1\\ 2 & 2 & 0\end{pmatrix}$ a $c = \begin{pmatrix}3\\ 1\\ -2\end{pmatrix}$.

Verify the correctness of your result x (resp. y) by computing the matrix product $Ax = b$ (resp. $By = c$).

Solution:

The solution for the system $Ax = b$ is the vector $x = (1, 0)^T$. We verify the correctness as follows:

$$
\begin{pmatrix}2 & 3\\ 1 & 2\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}2\cdot 1+3\cdot 0\\ 1\cdot 1+2\cdot 0\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix}.
$$

Solution for the system $By = c$ is the vector $y = (-1-t, t, 2)^T$, where $t \in \mathbb{R}$. We verify the correctness as follows:

$$
\begin{pmatrix}1 & 1 & 2\\ 1 & 1 & 1\\ 2 & 2 & 0\end{pmatrix}\begin{pmatrix}-1-t\\ t\\ 2\end{pmatrix} = \begin{pmatrix}1\cdot(-1-t)+1\cdot t+2\cdot 2\\ 1\cdot(-1-t)+1\cdot t+1\cdot 2\\ 2\cdot(-1-t)+2\cdot t+0\cdot 2\end{pmatrix} = \begin{pmatrix}3\\ 1\\ -2\end{pmatrix}.
$$

Problem 3. Prove or disprove whether for all matrices A, B, C and the zero matrix $\mathbf{0}$ of the same order and real numbers $\alpha, \beta \in \mathbb{R}$, it holds that:

(a) $A + (B + C) = (A + B) + C$

(b) $A + B = B + A$

(c) $A + \mathbf{0} = A$

(d) $\alpha(\beta A) = (\alpha\beta)A$

(e) $\alpha(\beta A) = \beta(\alpha A)$

(f) $A + (-1)A = \mathbf{0}$

(g) $1A = A$

(h) $A(B + C) = AB + AC$

(i) $\alpha(A + B) = \alpha A + \alpha B$

(j) $(\alpha + \beta)A = \alpha A + \beta A$

(k) $\alpha A + \beta B = (\alpha + \beta)(A + B)$

(l) $(A^T)^T = A$

(m) $A^T A$ is symmetric

(n) $(A + B)^T = A^T + B^T$

(o) $(\alpha A)^T = \alpha(A^T)$

(p) $AI_n = A$

Solution:

(a) The statement is correct.

First, we verify that both sides of the identity are well defined and that they have the same dimension. The matrices A, B, C are of the same order, which we denote $m \times n$. On the left side, we have an addition $B + C$ of matrices of order $m \times n$ and we get a matrix of order $m \times n$, and we add it to the matrix of order $m \times n$. Thus, the left side of the identity is well defined and of order $m \times n$.

Similarly, it holds also for the right side that $(A + B) + C$ is well defined and of order $m \times n$.

Now, we show the component-wise equality of the two sides. For all $i \in [n]$ and $j \in [m]$, it holds that:

$$\begin{aligned}
[A + (B + C)]_{i,j} &= [A]_{i,j} + [(B + C)]_{i,j} \\
&= [A]_{i,j} + ([B]_{i,j} + [C]_{i,j}) \\
&= ([A]_{i,j} + [B]_{i,j}) + [C]_{i,j} \quad \text{(by asociativity of addition over } \mathbb{R}) \\
&= [(A + B)]_{i,j} + [C]_{i,j} \\
&= [(A + B) + C]_{i,j}.
\end{aligned}$$

(b) The statement is correct.

Similarly as above, we first verify that both sides of the identity are well defined and that they have the same dimension for all matrices A and B. We argue the component-wise equality of the two sides; for all $\in [n]$ and $j \in [m]$,:

$$\begin{aligned}
[A + B]_{i,j} &= [A]_{i,j} + [B]_{i,j} \\
&= [B]_{i,j} + [A]_{i,j} \qquad \text{(by commutativity of addition over } \mathbb{R}) \\
&= [B + A]_{i,j}
\end{aligned}$$

(c) The statement is correct.

(d) The statement is correct.

(e) The statement is correct.

(f) The statement is correct.

(g) The statement is correct.

(h) The statement is correct (it corresponds to distributivity of multiplication and addition). For both sides to be well defined, it must hold for the matrices A, B, C that:

$$A \in \mathbb{R}^{m \times n}, \ B, C \in \mathbb{R}^{n \times p}$$

(otherwise both sides are not defined). By the assumption, the matrices A, B, C are all of the same order and, thus, they have to all be square.

(i) The statement is correct.

(j) The statement is correct.

(k) The statement is incorrect.

We give a counterexample. We give $\alpha, \beta \in \mathbb{R}$ and matrices A, B, which satisfy the assumptions but violate the statement. For example,

$$\alpha = 2, \ \beta = 3, \ A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ a } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We get

$$\alpha A + \beta B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \neq \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} (\alpha + \beta)(A + B) \ .$$

(l) The statement is correct.
If A is of order $m \times n$ and A^T is of order $n \times m$ and $(A^T)^T$ is again of order $m \times n$. Thus, the right and left side are of the same order.
We argue the component-wise equality of the two sides; for all $\in [n]$ and $j \in [m]$,:

$$[(A^T)^T]_{i,j} = [A^T]_{j,i} = [A]_{i,j} \ .$$

(m) The statement is correct.
By definition, a matrix D is symmetric if $D = D^T$. Note that if A is of order $m \times n$ then A^T is of order $n \times m$. Thus, the product $A^T A$ is well defined and of order $n \times n$.
We can use the theorem about properties of matrix transposition which gives that for all matrices D, E of compatible dimensions (so that their product is defined), it holds that $(DE)^T = E^T D^T$. Thus, we get

$$\begin{aligned} (A^T A)^T &= A^T (A^T)^T && \text{(using } (DE)^T = E^T D^T\text{)} \\ &= A^T A. && \text{(by the previous claim)} \end{aligned}$$

(n) The statement is correct.

(o) The statement is correct.

(p) The statement is correct only if A is a matrix of order $m \times n$ pro arbitrary m and n defined by I_n. Otherwise, the left side is not defined.

Problem 4. Express the elementary row operations as matrix products, i.e., for each elementary row operation, find a matrix $E \in \mathbb{R}^{m \times m}$ such that EA is the result of applying the operation to matrix A for all matrices $A \in \mathbb{R}^{m \times n}$.

Solution:

(a) *Multiplying the i-th row by scalar $\alpha \neq 0$.* We can use the matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \alpha & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} .$$

We take the identity matrix and set the i-th diagonal element equal to α.

The correctness of the above matrix E follows from the definition of matrix product. For all $A \in \mathbb{R}^{m\times n}$, the product EA is a matrix of order $m \times n$. For all row indices $j \in [m]$ and column indices $k \in [n]$ it holds that:

$$\begin{aligned}
[EA]_{j,k} &= \sum_l E_{j,l}A_{l,k} \\
&= E_{j,j}A_{j,k} && (E_{j,l} \neq 0 \text{ only for } l = j) \\
&= \begin{cases} A_{j,k} & \text{for } j \neq i \\ \alpha A_{j,k} & \text{for } j = i \end{cases} && (\text{substituting the values from } E_{j,j})
\end{aligned}$$

Therefore, the matrix EA has each row equal to the corresponding row of the matrix A – except for the i-th row, which is equal to the i-th row of A multiplied by α.

(b) *Swapping the i-th and the j-th row.* We can use the matrix

$$E = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}.$$

E is simply the identity matrix the i-th row and the j-th row swapped.

The correctness of the above matrix E follows from the definition of matrix product. For all $A \in \mathbb{R}^{m\times n}$, the product EA is a matrix of order $m \times n$. For all row indices $j \in [m]$ and column indices $k \in [n]$ it holds that:

$$\begin{aligned}
[EA]_{k,l} &= \sum_{\ell'} B_{k,\ell'}D_{\ell',l} \\
&= \begin{cases} E_{k,k}A_{k,l} & \text{for } k \neq i,j \\ E_{k,i}A_{i,l} & \text{for } k = j \\ E_{k,j}A_{j,l} & \text{for } k = i \end{cases} && (\text{for all other values } \ell',\ E_{k,\ell'} = 0) \\
&= \begin{cases} A_{k,l} & \text{for } k \neq i,j \\ A_{i,l} & \text{for } k = j \\ A_{j,l} & \text{for } k = i \end{cases} && (\text{substitung the values from } E)
\end{aligned}$$

Therefore, the matrix EA has each row equal to the corresponding row of the matrix A – except for the i-th row and the j-th row, which are swapped.

(c) *Adding an α-multiple of the i-th row to the j-th row, where $i \neq j$.* We can use the matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \alpha & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} .$$

We take the identity matrix and change the element at position i, j from 0 to α. The correctness of the above matrix E follows from the definition of matrix product. For all $A \in \mathbb{R}^{m \times n}$, the product EA is a matrix of order $m \times n$. For all row indices $j \in [m]$ and column indices $k \in [n]$ it holds that:

$$\begin{aligned} [EA]_{k,l} &= \sum_{\ell'} E_{k,\ell'} A_{\ell',l} \\ &= \begin{cases} E_{k,k} A_{k,l} & \text{for } k \neq i \\ E_{k,k} A_{k,l} + E_{k,j} A_{j,l} & \text{for } k = i \end{cases} && \text{(for all other values of } m,\ E_{k,\ell'} = 0) \\ &= \begin{cases} A_{k,l} & \text{for } k \neq i \\ A_{k,l} + \alpha A_{j,l} & \text{for } k = i \end{cases} && \text{(substituting the values from } E) \end{aligned}$$

Therefore, the matrix EA has each row equal to the corresponding row of the matrix A – except for the j-th row, which is the sum of the α-multiple of the i-th row and the j-th row.

Problem 5. Give a non-symmetric matrix A and a symmetric matrix B such that their product does not commute, i.e., such that $AB \neq BA$.

Is the product of symmetric matrices commutative?

<u>Solution</u>:
For the first part, we can use the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

We get

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = BA .$$

The statement does not hold even for symmetric matrices, as exemplified by the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

We get

$$AB = \begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}2 & 1 & 3\\ 4 & 2 & 5\\ 5 & 3 & 6\end{pmatrix} \neq$$

$$\neq \begin{pmatrix}2 & 4 & 5\\ 1 & 2 & 3\\ 3 & 5 & 6\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{pmatrix} = BA.$$

Problem 6. Prove or disprove the following statements:

(a) For all $A, B \in \mathbb{R}^{n\times n}$, if A is symmetric and commutes with B then A commutes also with B^T.

(b) For all $A, B \in \mathbb{R}^{n\times n}$, if A commutes with B then A commutes with B^T.

Solution:

(a) The statement is correct $AB^T = A^T B^T = (BA)^T = (AB)^T = B^T A^T = B^T A$.

(b) The statement is incorrect. The counterexample are matrices

$$A = \begin{pmatrix}0 & 1 & 2\\ 0 & 3 & 4\\ 0 & 0 & 0\end{pmatrix} \text{ and } B = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$