NMAI057 – Linear algebra 1

Tutorial 1

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Problem 1. List as many ways as possible to specify a line in space. Discuss the assumptions and limitations of individual approaches.

Solution:
There are many possibilities:

- *A point on the line with the direction vector.* Any point on the line together with an arbitrary non-zero vector in the direction of the line.
- *Two points on the line.* Arbitrary distinct points on the line.
- *Two linear equations.* Two equations defining two distinct planes, i.e., one equation is not a multiple of the other. Moreover, their normals must be non-zero in order to describe a plane.

Problem 2. Find a linear equation defining the plane given by the point $[3, 2, 1]$ and the slopes $(1, 1, 1)$, $(2, -1, 0)$.

Solution:
The normal of the plane can be found, for example, using the vector product (a.k.a. the cross product) of the two slopes $(1, 1, 1) \times (2, -1, 0) = (1, 2, -3)$. Thus, the equation defining the plane is of the form $x_1 + 2x_2 - 3x_3 = d$. The constant term d can be found using the known point on the plane as $d = 1 \cdot 3 + 2 \cdot 2 - 3 \cdot 1 = 4$. We conclude that the equation is

$$x_1 + 2x_2 - 3x_3 = 4.$$

In case we do not want to rely on the knowledge of cross product, we can find the normal in the general form (a, b, c) and the corresponding equation in the form $ax_1+bx_2+cx_3 = d$. Since the plane contains the point $[3, 2, 1]$, we get an equation

$$3a + 2b + c = d.$$

Since the slopes $(1, 1, 1), (2, -1, 0)$ must be orthogonal to the normal, we get additional two equations

$$a + b + c = 0, \quad 2a - b = 0.$$

And we have a system of three linear equations with four unknowns, which is no surprise as the sought equation describing the plane is not unique – any multiple of the equation would work. By solving the system, we get a parametric solution $a = t$, $b = 2t$, $c = -3t$, $d = 4t$ for $t \in \mathbb{R}$. We can choose, for example, $t = 1$ (or any other non-zero $t \in \mathbb{R}$) and we get (again) the equation

$$x_1 + 2x_2 - 3x_3 = 4.$$

Try verifying the equation using the point and the slopes!

Problem 3. Find a parametric description of the plane defined by the linear equation $2x_1 + 3x_2 + x_3 = 4$.

Solution:

We find one point of the plane by choosing the value of two components arbitrarily and calculating the remaining one. For example, let's choose $x_2 = x_3 = 0$ and from the equation, we get $x_1 = 2$. Thus, one point on the plane is $[2, 0, 0]$.

We get the slopes as two distinct vectors (they must not be multiples of each other) othogonal to the normal $(2, 3, 1)$. It is easy to see that such vector are, for example, $(0, 1, -3)$ a $(1, 0, -2)$. This follows as any slope (x, y, z) must be orthogonal to the normal, i.e., $2x + 3y + z = 0$. We find two distinct solution of this equation which are not multiples of each other (so that we get two distinct directions for the slopes). For example, we set $x = 0, y = 1$ and compute $z = -3$, and for the second solution we set $x = 1, y = 0$ and compute $z = -2$.

Problem 4. Find a parametric description of the line given by the two equations:

$$x_1 + 3x_2 + x_3 = 2, \quad 2x_1 + 5x_2 + x_3 = 3.$$

Solution:
We simply solve the system and describe the solution using the parameter $t \in \mathbb{R}$.

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 2 & 5 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & -1 & -1 & -1 \end{array}\right)$$

We can set x_3 to be a real parameter and express x_2 as $x_2 = 1 - x_3$. Substituting into the first equation, we get $x_1 = 2 - 3(1 - x_3) - x_3 = -1 + 2x_3$. We get the parametric description of the line as all the points $[-1 + 2x_3, 1 - x_3, x_3] = [-1, 1, 0] + x_3(2, -1, 1)$ for $x_3 \in \mathbb{R}$. In other words, the line passing through the point $[-1, 1, 0]$ with a slope $(2, -1, 1)$.

Problem 5. Find two equations defining the line $[3, 2, 1] + t(1, -1, 1)$, where $t \in \mathbb{R}$.

Solution:
Both equations must be satisfied by the point $[3, 2, 1]$ and the normal must be orthogonal to the slope $(1, -1, 1)$. Moreover, the resulting equations must define distinct planes, i.e., they must not be multiples of each other.

Let's choose a normal $(1, 1, 0)$, which is orthogonal to the slope. The corresponding equation for this normal is $x_1 + x_2 = d$ and, using the known point $[3, 2, 1]$ on the point, we compute $d = 5$. Now, we choose a different normal $(0, 1, 1)$, which is also orthogonal to the slope. The corresponding equation for this normal is $x_2 + x_3 = d'$ and, using the known point $[3, 2, 1]$ on the point, we compute $d' = 3$. Thus, the sought equations are, for example, $x_1 + x_2 = 5$, $x_2 + x_3 = 3$.

Note that the solution is not unique. In the second step, we could have chosen as the normal the vector $(1, 0, -1)$, which leads to the equation $x_1 - x_3 = 2$. Thus, the equations $x_1 + x_2 = 5$, $x_1 - x_3 = 2$ also constitute a valid solution.

Finaly, note that we do not need more that two equations that are not multiples of each other. If we would add to the equations $x_1 + x_2 = 5$, $x_2 + x_3 = 3$ the equation $x_1 - x_3 = 2$ then the system $x_1 + x_2 = 5$, $x_2 + x_3 = 3$, $x_1 - x_3 = 2$ also defines the given line. However, the third equation is redundant. Indeed, you can easily verify that the the last equation is the difference of the first and the second.

Problem 6. Determine all possible mutual positions of two lines in the space $\mathbb{R}^3$. Next, describe how the positions can be determined if both lines are defined parametrically or by equations.

Solution:
Possible positions of the lines:

- *Parallel.*

  Parametrically: The direction vector of one line is a multiple of the direction vector of the other line, but the lines do not have an intersection.

  Usin equations: All normals are in the same plane, i.e., each normal can be expressed as the sum of multiples of the normals of the equations of the second line. Furthermore, no point satisfies all the equations at once.
- *Identical.*

  Parametrically: The direction vector of one line is a multiple of the direction vector of the other line and in addition the lines have an intersection.

  Equation: All normals are in one plane, i.e., each normal can be expressed as the sum of multiples of the normals of the equations of the second line. Furthermore, at least one point satisfies all equations at once.
- *Intersecting.*

  Parametrically: The direction vector of one line is not a multiple of the direction vector of the other line, and the lines have an intersection.

  Equation: We cannot express at least one normal as the sum of multiples of the normals of the equations of the second line. Furthermore, at least one point satisfies all equations at once.
- *Non-intersecting.*

  Parametrically: The direction vector of one line is not a multiple of the direction vector of the other line and the lines do not have an intersection.

  Equation: We cannot express at least one normal as the sum of multiples of the normals of the equations of the second line. Furthermore, no point satisfies all the equations at once.

Problem 7. Determine the relative position of the two lines given by a point and a slope

$$p : [1, 5, 3],\ (1, -2, -2), \quad q : [3, 1, -1],\ (-1, 2, 2).$$

Solution:
Because the slopes are multiples of each other, the lines are either parallel or identical. We can easily verify that point $[1, 5, 3]$ of the line p lies also on the line q since $[1, 5, 3] = [3, 1, -1] + t(-1, 2, 2)$ for $t = 2$. Thus, the lines are identical.

Problem 8. Interpolate a quadratic function through the points $[1, 1]$, $[2, 2]$, $[3, 7]$.

Solution:
A quadratic function has the form $y = ax^2 + bx + c$. By substituting the three known points, we get a linear system of three equations in three unknowns

$$a + b + c = 1,\ 4a + 2b + c = 2,\ 9a + 3b + c = 7,$$

with the solution $a = 2$, $b = -5$, $c = 4$. Thus, the sought quadratic function is

$$y = 2x^2 - 5x + 4.$$