

Probabilistic techniques - tutorials

Classwork 2 – Expectations and method of alterations

1. Show that there is a two coloring of the edges of K_n with at most $\binom{n}{a} 2^{1-\binom{a}{2}}$ monochromatic K_a .

Solution: For $S \in \binom{V}{a}$ let be I_S be the indicator that $K_n[S]$ is monochromatic. Then

$$\mathbb{E}[\# \text{ monochromatic } K_a] = \mathbb{E} \left[\sum_S I_S \right] = \sum_S \mathbb{E}[I_S] = \binom{n}{a} \Pr[K_n[S] \text{ monochromatic}] = \binom{n}{a} 2^{1-\binom{a}{2}}.$$

Thus, there is a coloring with as few as $\binom{n}{a} 2^{1-\binom{a}{2}}$ monochromatic copies of K_a .

2. Let M be an $n \times n$ matrix with entries uniformly independent chosen from $\{-1, 1\}$. Determine $\mathbb{E}[\det(M)]$.

Solution: We have

$$\mathbb{E}[\det(M)] = \mathbb{E} \left[\sum_{\pi \in S_n} \text{sgn}(\pi) \prod_i M_{i, \pi(i)} \right] = \sum_{\pi \in S_n} \text{sgn}(\pi) \prod_i \mathbb{E}[M_{i, \pi(i)}] = 0.$$

3. Let $n \geq 2$, $H = (V, E)$ an n -uniform hypergraph with $|E| = 4^{n-1}$ edges. Show that there is a coloring of V by four colors such that no edge is monochromatic.

Solution: For $e \in E$ let be I_e be the indicator that e is monochromatic. Then

$$\mathbb{E}[\# \text{ monochromatic edges}] = \mathbb{E} \left[\sum_e I_e \right] = |E| \Pr[e \text{ monochromatic}] = 4^{n-1} \cdot 4^{1-n} = 1.$$

Since there are events attaining greater value than 1 (e.g. monochromatic V), there are events fewer monochromatic edges than 1, i.e. 0.

4. Denote by $R(\cdot, \cdot)$ the Ramsey numbers. Fix $k \in \mathbb{N}$. Prove that for any integer n , it holds that $R(k, k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$. Can you write the expression on the right hand side in terms of k ?

Solution: (Alon-Spencer Theorem 3.1.1) Take a random coloring of K_n and remove one vertex from each monochromatic copy of K_k . There is a coloring with at most $\binom{n}{k} 2^{1-\binom{k}{2}}$ copies. Thus, there is a graph with at least $n - \binom{n}{k} 2^{1-\binom{k}{2}}$ vertices without monochromatic copy of K_k .

Choosing n as a suitable function of k , you can prove

$$R(k, k) > \frac{1}{e} (1 + o(1)) k 2^{k/2}.$$

5. Fix a graph F and define $\text{ex}(n, F)$ to be the maximal number of edges of an n -vertex graph that does not contain F as a subgraph. Show that for any F on $k > 2$ vertices with at least $2k - 3$ edges it holds $\text{ex}(n, F) = \Omega(n^{3/2})$.

Solution: Consider $G(n, p)$. To ensure that the graph does not contain F , we remove one edge from each k tuple containing at least $2k - 3$ edges. The expected number of such copies is at most $n^k p^{2k-3}$. Let X stand for the number of edges of the resulting graph. Then, by setting $p = cn^{-1/2}$, we have

$$\mathbb{E}[X] \geq \binom{n}{2} cn^{-1/2} - n^k c^{2k-3} n^{-1/2(2k-3)} \geq \frac{c}{3} n^{3/2} - c^{2k-3} n^{3/2} = \left(\frac{1}{3} - c^{2k-4} \right) cn^{3/2}.$$

We can set $c = 1/3$.

6. A dominating set of an undirected graph $G = (V, E)$ is a set $U \subseteq V$ such that every vertex $v \in V \setminus U$ has at least one neighbor in U . Let $G = (V, E)$ be a graph on n vertices, with minimum degree $\delta > 1$. Then G has a dominating set of at most $n \frac{1+\ln(\delta+1)}{\delta+1}$ vertices.

Solution: Let D be a dominating set created as follows. Put $v \in A$ independently with probability p . Then, let B be the set of vertices from $V \setminus A$ that are not dominated by A . Set $D = A \cup B$. We have $\mathbb{E}[A] = np$ and $\mathbb{E}[B] \leq n(1-p)^{\delta+1}$. Thus the expected size of D is

$$\mathbb{E}[D] = \mathbb{E}[A] + \mathbb{E}[B] \leq np + n(1-p)^{\delta+1} \leq np + ne^{-p(\delta+1)}.$$

By setting $p = \frac{\ln(\delta+1)}{\delta+1}$, we get the result in expectation. Thus, there exists such a set D .

7. Let $G = (V, E)$ be a graph on n vertices with minimum degree $\delta > 1$. Prove that there is a partition of V into two disjoint sets A and B such that $|A| \leq \mathcal{O}(n \frac{\ln(\delta)}{\delta})$ and each vertex in B has at least one neighbor in A and at least one neighbor in B .

Solution: We say that a vertex $v \in V$ is over-dominated by a set $S \subset V, v \notin S$ if $N(v) \subset S$. First put $v \in X$ independently with probability p . Let Y be the vertices over-dominated by X and set $A_0 = X \cup Y$. It remains to solve non-dominated vertices (i.e. vertices with no neighbours in A_0), let us call them Z_0 . We iteratively build sets A_i and Z_i . At the step i , for each vertex $v \in Z_i$ we proceed as follows: let $A_{i+1} = A_i \cup \{v\}$ if it does not create a new over-dominated vertex. Otherwise, let $A_{i+1} = A_i \cup \{u\}$ instead, where u is the neighbor of v to be over-dominated. Update the set Z_i to Z_{i+1} by removing any remaining dominated vertices.

Observe that $|Z_i|$ always decreases, thus at the end of the process, we have $|A| \leq |X| + |Y| + |Z_0|$. Therefore, in expectation

$$\mathbb{E}[|A|] \leq np + n(1-p)p^\delta + n(1-p)^{\delta+1} \leq n(2p + e^{-p(\delta+1)}),$$

which for $p = \frac{\ln(\delta)}{\delta+1}$ simplifies as follows

$$\mathbb{E}[|A|] \leq n \left(2 \cdot \frac{\ln(\delta)}{\delta+1} + \frac{1}{\delta} \right) \leq n \left(2 \cdot \frac{\ln(\delta)}{\delta} + 2 \cdot \frac{\ln(\delta)}{\delta} \right) = 4n \frac{\ln(\delta)}{\delta}.$$