8. cvičení z PSt

1.

$$(a)\ P(X \in [0,1]) = \int_0^1 f_X(t)\,dt.$$
$$(b)\ P(X > 0) = \int_0^\infty f_X(t)\,dt.$$
$$(c)\ P(X < 0) = \int_{-\infty}^0 f_X(t)\,dt.$$

2. Nechť bod zlomu $U \sim U(0,1)$. Delší část má délku

$$X = \max(U, 1-U).$$

(a) Pro $x \in [1/2, 1]$:

$$F_X(x) = P(X \leq x) = P(1-x \leq U \leq x).$$

Odtud

$$F_X(x) = x - (1-x) = 2x - 1, \quad x \in [1/2, 1].$$

Pro $x < 1/2$ je $F_X(x) = 0$, pro $x \geq 1$ je $F_X(x) = 1$.

Hustota:

$$f_X(x) = 2, \quad x \in (1/2, 1).$$

(b)

$$E(X) = \int_{1/2}^1 x \cdot 2\,dx = 2 \cdot \frac{x^2}{2}\Big|_{1/2}^1 = 1 - \frac{1}{4} = \frac{3}{4}.$$

3. Nechť Z je definovaná jako:

$$Z = \begin{cases} X, & \text{s pravd. } p, \\ Y, & \text{s pravd. } 1-p. \end{cases}$$

(a)

$$F_Z(z) = P(Z \leq z) = pP(X \leq z) + (1-p)P(Y \leq z) = pF_X(z) + (1-p)F_Y(z).$$

(b) Zderivováním dostaneme

$$f_Z(z) = pf_X(z) + (1-p)f_Y(z).$$

4.

$$(a)\ \int_1^\infty \frac{1}{t^2}dt = \left[-\frac{1}{t}\right]_1^\infty = 1.$$

$$(b)\ E(X) = \int_1^\infty \frac{1}{t}dt = \infty.$$

$$(c)\ F(x) = \begin{cases} 0, & x < 1, \\ 1 - \frac{1}{x}, & x \geq 1. \end{cases}$$

$$(d)\ P(2 \leq X \leq 3) = F(3) - F(2) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

$$(e)\ Y = \frac{1}{X}.$$

$$F_Y(y) = P(Y \leq y) = P\left(X \geq \frac{1}{y}\right) = 1 - F\left(\frac{1}{y}\right) = y, \quad y \in (0,1).$$

$$(f)\ f_Y(y) = 1, \quad y \in (0,1).$$

5.

$$M = \min(X_1, \ldots, X_n).$$

Pro $t \geq 0$:

$$P(M > t) = P(X_1 > t, \ldots, X_n > t) = \prod_{i=1}^{n} P(X_i > t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-(\lambda_1 + \cdots + \lambda_n)t},$$

kde druhá rovnost je z nezávislosti. Tedy

$$F_M(t) = 1 - e^{-(\lambda_1 + \cdots + \lambda_n)t},$$

což je exponenciální rozdělení s parametrem $\lambda_1 + \cdots + \lambda_n$.

6. Nechť $Y = \max(U_1, \ldots, U_n)$, kde $U_i \sim U(0,1)$.

(a)

$$F_Y(y) = P(U_1 \leq y, \ldots, U_n \leq y) = y^n, \quad y \in [0,1].$$

(b)

$$f_Y(y) = ny^{n-1}, \quad y \in (0,1).$$

(c)

$$E(Y) = \int_0^1 y \cdot ny^{n-1} dy = n \int_0^1 y^n dy = \frac{n}{n+1}.$$

7. (a) Konvoluce:

$$f_{X+Y}(t) = \int_0^1 f_X(s) f_Y(t-s)\, ds.$$

Výpočet integrálu:

1) $t \in [0,1]$:

$$f_S(t) = \int_0^t 1\, ds = [s]_0^t = t - 0 = t.$$

2) $t \in [1,2]$:

$$f_S(t) = \int_{t-1}^1 1\, ds = [s]_{t-1}^1 = 1 - (t-1) = 2 - t.$$

9.

(a) $f_X(x) = \int_0^\infty e^{-x-y}dy = e^{-x}, \quad x > 0.$

$f_Y(y) = e^{-y}, \quad y > 0.$

(b) $F_X(x) = 1 - e^{-x}, \quad x > 0.$

$F_{X,Y}(x, y) = \int_0^x \int_0^y e^{-s-t}dtds = (1 - e^{-x})(1 - e^{-y}).$

(c) $f_{X,Y}(x, y) = f_X(x)f_Y(y) \Rightarrow X, Y$ nezávislé.

(d) $P(X + Y \leq 1) = \int_0^1 \int_0^{1-x} e^{-x-y}dydx = \int_0^1 e^{-x}(1 - e^{-(1-x)})dx.$

Úpravou: $= 1 - 2e^{-1}.$

$P(X > Y) = \int_0^\infty \int_0^x e^{-x-y}dydx = \frac{1}{2}.$