Suolotions for Tutorial Sheet 1 - 7.10.2024

Exercise 0. For the lower bound, we use the upper bound for the integral.

1.
$$\int_1^n \ln x \, dx \leq (\ln n + \cdots + \ln 2) \times 1 = \ln(n!)$$

For the upper bound, we use the lower bound for the integral.

2.
$$(\ln(n-1) + \cdots + \ln 1) \cdot 1 = \ln(n-1)! \leq \int_1^n \ln(x) \, dx$$

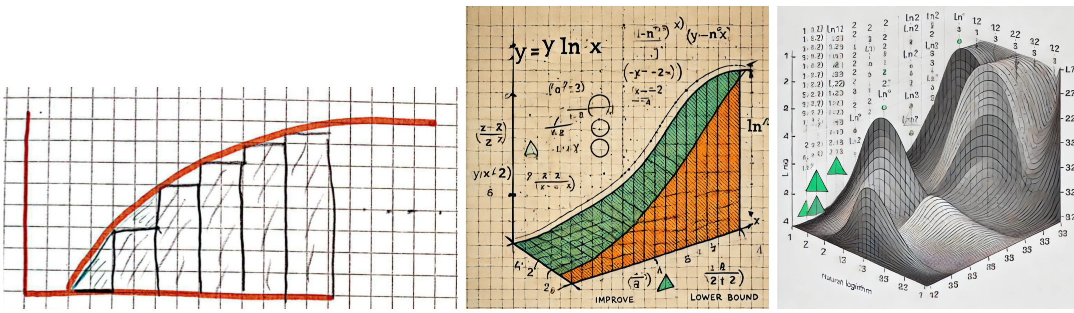

Figure 1: You can see green triangles in the left picture, the right pictures are what I received when I asked GPT.

We can improve the upper bound which means the second inequality by adding green triangles in the shape.

$$\underbrace{\frac{1}{2}\left((\ln n - \ln(n-1)) + \cdots + (\ln 2 - \ln 1)\right)}_{\text{green triangles}} + \ln(n-1)! \leq \int_1^n \ln(x)$$

We already know that:(take a derivative from the right side and you get the left side)

$$\int_1^y \ln(x) = y\ln(y) - y + 1$$

So we have

$$\frac{1}{2}\ln n + \ln((n-1)!) \leq n\ln(n) - n + 1$$

So we have

$$\ln(n!) \leq (n - \frac{1}{2})\ln(n) - n + 1$$

So now we compute e power both sides and we get the answer

Exercise 1. Define string w_A as follow, put a in position i if $x_i \in A$ and b otherwise.
Now we have a bijection between subsets of X and stings from $\{a, b\}^n$. so the number of different subsets are 2^n

Exercise 2. Define string $w_{A \subset B}$ as follow, put a in position i if $x_i \in A$ and b if it was in $B - A$ and c otherwise.
Now we have a bijection between pairs($A \subset B$) of subsets of X and stings from $\{a, b, c\}^n$. so the number of different pairs of $A \subset B$ subsets are 3^n.

Exercise 3. for a divisor d there is also devisor $\frac{n}{d}$ so in this way we pair of all the devisors, except $\sqrt{n}$. If we are among the devisors it is the only one we can't pair up that's it's the only way that we get an odd number of devisors. So it gives us the answer.

Exercise 4. Fix 1 now it turns to the arrangement of all other members in a line that is $n-1!$

Exercise 5. 5×3. if we have k cycles with length $c_1, \cdots, c_k$ the answer is the smallest common multiple of all of them.

$$LCM(c_1, \cdots, c_k)$$

Exercise 6. First, we use the fact that the number of different solutions for $a_1 + \cdots + c_n = m \quad c_i \in \{0\} \cup \mathbb{N}$ that is the same questions of arrangement of $n-1$ number of $+$ and m number of 1 we know the answer is

$$\binom{m+n-1}{n-1}.$$

So now here monotonicity means $a_i := f(i) - f(i-1)$ is from $\{0\} \cup \mathbb{N}$ for $2 \leq i \leq n$. and also $a_1 := f(1) - 1$ for the condition on the board set. so for every monotone function, I will get a set of a_is such that:

$$(1 + a_1) + \cdots + (a_n) \leq n \quad , a_i \in \{0\} \cup \mathbb{N}$$

and vice-versa. so I change the problem to

$$(a_1) + \cdots + (a_{n+1}) = n - 1 \quad , a_i \in \{0\} \cup \mathbb{N}$$

and there is a bijection between any of these two answers. and we know the answer to this now:

$$\binom{2n-1}{n}.$$

Exercise 7. I want to choose $r+1$ members from the set: $\{1, \cdots, n+1\}$ at first, choose the maximum of the chosen ones and then choose all the others. so if I choose $n+1$ I should choose all the others(r remained) from $\{1, \cdots n\}$ and $\ldots$ till when I chose $r+1$ as maximum the I should choose r others ones from $\{1, \cdots r\}$ and the maximum can not be less that that.

Exercise 8. We are looking for a different

$$x_1^{a_1} \cdots x_m^{a_m}$$

that we can have such that

$$a_1 + \cdots + a_m = n \quad , a_i \in \{0\} \cup \mathbb{N}$$

so the answer is

$$\binom{n+m-1}{n}.$$

Exercise 9.

$$1000n \quad n \log n \quad n\sqrt{n} \quad \frac{1}{2}n(n+1) \quad 1.1^n$$

Exercise 10.

$$n\sqrt{n} \quad \binom{2n}{5} \quad n^5 \quad \binom{2n}{n} \quad (\sqrt{n})^n \quad n! \quad n^n$$

Exercise 11. Take the derivate and solve when it reaches 0. The answer is $x = 0$. that's the minimum thing that we have for $e^x - x - 1$ in all other domain this function is greater than that(0).

Exercise 12. instead of induction use the fact that $\ln x' = \frac{1}{x}$ and get the answer. you can use also **exercise 11** for induction but I am not going to solve that.