Tutorial Sheet 9 - 09.12.2024

Double counting

Exercise 1. In a square(4 vertices), there are n vertices in general position, we connect the vertices with edges until the polygon is divided into triangles. The number of triangles is always $T = 2n + 2$, and the number of edges is always $E = 3n + 1$.

Exercise 2. A 10×10 table is filled with ones and zeros such that the sums of any two rows are different, and the sums of all columns are the same. What are the sums in the columns?

Exercise 3. Let G be a planar graph drawn so that all faces are triangles. Assume the vertices of G are colored with three colors (not necessarily a proper coloring, i.e., an edge may have both endpoints of the same color). Show that the number of triangular faces, on which all three colors are used, is even.

Exercise 4. Prove that a 5-regular graph with 20 vertices contains a four-cycle C_4.

Exercise 5. Twenty students wrote an exam consisting of four questions. For every pair of students, there was a question that both solved correctly. Prove that there is a question solved by at least half of the students.

Exercise 6. In a chess tournament of n players, every player plays against every other player exactly once (no draws). For a player $i = 1, 2, \dots, n$, let w_i and l_i denote the number of wins and losses of the player, respectively.

(a) Prove that $\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} l_i$.

(b) Prove that $\sum_{i=1}^{n} w_i^2 = \sum_{i=1}^{n} l_i^2$.

Exercise 7. In an $n \times n$ grid, write the number of rectangles (including squares) each cell belongs to. Prove that the sum of all such numbers equals the square of the sum of the integers up to n^2.

Exercise 8 [Erdős–Ko–Rado theorem]. Let $n \geq 2r$ be natural numbers. From the set $[n] = \{1, 2, \dots, n\}$, we select r-element subsets such that every pair of subsets has a non-empty intersection. Prove that the maximum number of such subsets is $\binom{n-1}{r-1}$.

Exercise 9. Count the number of spanning trees in the following graphs:

(a) C_n, a cycle on n vertices.

(b) $C_m \oplus e$, a cycle with an additional edge e.

(c) $K_n - e$, the complete graph K_n minus one edge e.

(d) $K_n \div e$, the complete graph K_n with one subdivided edge e.

(e) (Bonus) A complete bipartite graph $K_{a,b}$.

Connectivity

Exercise 10. What are vertex connectivity and edge connectivity of a n-dimension Hypercube?

Notes:

Theorem (Sperner's theorem). Let n be a non-negative integer, and let X be an n-element set. Then any antichain in $\mathcal{P}(X), \subseteq$ has at most $\binom{n}{\lfloor n/2 \rfloor}$ elements. Furthermore, this bound is tight, that is, there exists an antichain in $\mathcal{P}(X), \subseteq$ that has precisely $\binom{n}{\lfloor n/2 \rfloor}$ elements.

Theorem (Mantel's theorem). The maximum number of edges in an n-vertex triangle-free graph is $\lfloor n^2/4 \rfloor$.

Theorem (Maximal C_4-free). Let n be a positive integer. Any graph on n vertices that does not contain C_4 as a subgraph has at most $\frac{1}{2}\left(n + n^{3/2}\right)$ edges.

Theorem (Recursion). $T(G) :=$ The number of spanning trees of graph G,

$$T(G) = T(G - e) + T(G : e).$$

$\mathbf{G - e}$: Graph G minus edge e,
$\mathbf{G : e}$: Graph G after contraction[1] on edge G.

Theorem (Cayley's formula).

$$T(K_n) = n^{n-2}.$$

An **antichain** in $(P(X), \subseteq)$ is any set $\mathcal{A}$ of subsets of X such that for all distinct $A_1, A_2 \in \mathcal{A}$, we have that $A_1 \nsubseteq A_2$ and $A_2 \nsubseteq A_1$

[1] In graph theory, an edge contraction is an operation that removes an edge from a graph while simultaneously merging the two vertices that it previously joined. and we may have multiple edges in this definition