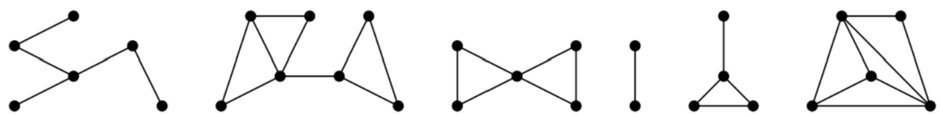
Tutorial Sheet 8 - 02.12.2024

Exercise 1. Determine the edge and vertex connectivity of the following graphs.



Exercise 2. For every $k \geq 1$, find a graph that has vertex connectivity 1 and edge connectivity k.

Exercise 3. Find an example of a graph G and a vertex v such that removing v from G:

1. Decreases the edge connectivity of G by 10.
2. Increases the edge connectivity of G by 10.
3. Increases the vertex connectivity of G by 10.
4. *Bonus:* By how much can the vertex connectivity decrease the most by removing a single vertex?

Exercise 4. An oriented graph is said to be strongly connected if it is possible to reach any vertex from any other vertex via a directed path.

1. Prove that if G is vertex 2-connected, then it has an edge orientation that makes it strongly connected.
2. Prove that the converse implication does not hold.

Exercise 5. Let G be a critically 2-connected graph, meaning that G is vertex 2-connected, but none of the graphs $G - e$ for any $e \in E(G)$ is vertex 2-connected.

1. Prove that at least one vertex of G has degree 2.
2. For every $n \geq 2$, provide an example of a critically 2-connected graph with a vertex of degree at least n.

Exercise 6. Prove that every vertex k-connected graph with at least $2k$ vertices contains a cycle of length at least $2k$.

Exercise 7. Let G be any graph on $2n$ vertices, where every vertex has degree at least n. Prove that G is edge n-connected.

An **edge cut** in a graph $G = (V, E)$ is a set of edges $F \subseteq E$ such that the graph $(V, E \setminus F)$ is disconnected. The **edge connectivity** $\mathbf{k_e(G)}$ of a graph G with $n \geq 2$ vertices is the size of the smallest edge cut in G. Similarly, a **vertex cut** in a graph G is a set of vertices $A \subseteq V$ such that the graph $(V \setminus A, E \cap \binom{V \setminus A}{2})$ is disconnected.

The **vertex connectivity** $\mathbf{k_v(G)}$ of a graph G is defined as $n-1$ if G is a complete graph K_n with $n \geq 2$, and as the size of the smallest vertex cut in G otherwise.

A graph G is edge t-connected for $t \in \mathbb{N}_0$ if $k_e(G) \geq t$, and vertex t-connected if $k_v(G) \geq t$. (For $n = 1$, we define $k_e(K_1) = k_v(K_1) = 1$.)

Lemma (Edge connectivity is larger than vertex connectivity). For every graph G, $k_v(G) \leq k_e(G)$.

Lemma (Ear lemma). A graph is vertex 2-connected if and only if it can be obtained from a cycle by attaching ears.

Theorem (Menger's theorem). For any graph G with $n \geq 2$ vertices and a natural number $t \in \mathbb{N}$, the following holds:

- $k_e(G) \geq t$ if and only if there exist at least t edge-disjoint paths between every pair of distinct vertices.
- $k_v(G) \geq t$ if and only if there exist at least t vertex-disjoint paths between every pair of distinct vertices.