Tutorial Sheet 7 - 25.11.2024

1. (a) Does the set system $M_4 = (\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\})$, consisting of all 3-element subsets of the set $\{a, b, c, d\}$, have an SDR?

   (b) Does a similar set system M_5, consisting of all 3-element subsets of $\{a, b, c, d, e\}$, have an SDR?

   (c) Find a set system M_6 consisting of six 3-element subsets of $\{a, b, c, d, e, f\}$ that does not have an SDR.

2. Prove that the two versions of Hall's theorem are equivalent.

3. Let G be a k-regular bipartite graph ($k \geq 1$).

   (a) Prove that the parts of the graph G have the same size.

   (b) Prove that G has a perfect matching.

   (c) Prove that the edges of G can be colored with k colors so that the edges of each color form a perfect matching.

4. On an 8×8 chessboard, there are 24 rooks placed such that each row and each column contains exactly 3. Prove that it is possible to rearrange the rows and columns such that 8 of the rooks lie on the diagonal.

5. A standard deck of 52 cards is shuffled, and the cards are dealt into 13 piles of 4 cards each. Is it always possible to select one card from each pile so that the selected cards include one ace, one 2, one 3, ..., up to one king?

6. A Latin square of order n is an $n \times n$ grid filled with the numbers $1, 2, \ldots, n$, such that each number appears exactly once in each row and each column.

   (a) Prove that if the first two rows are filled with $1, 2, \ldots, n$ such that no number repeats in rows or columns (so far), the remaining cells can be completed to form a Latin square.

   (b) Prove that the number of Latin squares of order n is $\Omega((n!)^2)$.

7. Find an infinite set system that satisfies Hall's condition but does not have an SDR.

8. (*Putnam 2013*) In a chess tournament with $2n$ players and $2n - 1$ rounds, each player plays against every other player. No match ends in a draw. Arpad had a crystal ball and knew in advance who would win each match. Prove that he could choose one match to watch in each round such that he sees $2n - 1$ different players win.

9. (*124 cards*) Adam and Bara perform the following trick with a deck of 124 cards numbered $1, 2, \ldots, 124$. A spectator selects 5 cards and gives them to Adam. Adam hides one card and arranges the remaining four on a table in some order. Bara then determines the hidden card.

   (a) Prove that the trick cannot work reliably for more than 124 cards.

   (b) Prove that the trick can work reliably for 124 cards.

A **matching** in a graph $G = (V, E)$ is a set of disjoint edges from E. A matching is perfect if every vertex is an endpoint of one edge.

A **vertex cover** is a set $C \subseteq V$ such that for every $e \in E$, $e \cap C \neq \emptyset$.

For $A \subseteq V$, let $N_G(A)$ denote the set of neighbors of A, i.e., the set of vertices in $V \setminus A$ that are adjacent to at least one vertex in A.

Let X and I be finite sets. A set system on X is any $|I|$-tuple of subsets of X, i.e., $M = (M_i : i \in I)$, where $M_i \subseteq X$.

A **system of distinct representatives (SDR)** is an injective function $f : I \to X$ such that for every $i \in I$, $f(i) \in M_i$.

Theorem (Hall's for bipartite graphs). A bipartite graph with parts A and B has a matching saturating part A if and only if for every subset $A' \subseteq A$,

$$|N_G(A')| \geq |A'|.$$

Theorem (Hall's for set systems). A set system $M = (M_i : i \in I)$ has an SDR if and only if for every subset $J \subseteq I$,

$$\left| \bigcup_{j \in J} M_j \right| \geq |J|.$$

This condition is known as Hall's condition.

Theorem (Kőnig–Egerváry). In every bipartite graph, the size of a minimum vertex cover equals the size of a maximum matching.