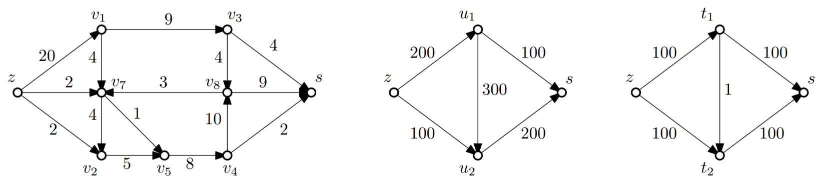
Tutorial Sheet 6 - 18.11.2024



Exercise 1 . Find the maximum flow in the following network using the Ford–Fulkerson algorithm. Also, find the minimum cut capacity and verify that the given flow has the maximum size.

Exercise 2. Find

1. A network (and a sequence of augmenting paths used) where the Ford–Fulkerson algorithm does not yield the correct result if only directed augmenting paths are allowed.
2. A sequence of networks (and sequences of augmenting paths used) where the Ford–Fulkerson algorithm exhibits exponential time complexity (in terms of the number of bits required to store the graph and capacities).

Exercise 3. When searching for a maximum flow, edges are constrained by their capacities. However, sometimes we may also need to assign capacities to vertices, such that "no more than x liters of fluid can pass through vertex v per unit time." How can we find a maximum flow that satisfies this condition?

Exercise 4. Show that the problem of finding a maximum flow in a network with multiple sources and sinks can be reduced to the case with a single source and a single sink.

Exercise 5. Prove that the number of maximum flows in any network is either one or infinite.

A *network* is an ordered quadruple (G, s, t, c), where $G = (V, E)$ is a directed graph ($E \subseteq V \times V$), s and t are two distinct vertices of the graph G (called the source and the sink), and $c : E \to \mathbb{R}_0^+$ is a capacity function assigning a non-negative real number to the edges.

A *flow* in the network is any function $f : E \to \mathbb{R}$ satisfying $0 \leq f(e) \leq c(e)$ for every edge $e \in E$ and

$$\forall u \in V \setminus \{s, t\} : \qquad \sum_{v:(u,v)\in E} f(u, v) = \sum_{v:(v,u)\in E} f(v, u)$$

The *size of the flow* is

$$w(f) = \sum_{v:(s,v)\in E} f(s, v) - \sum_{v:(v,z)\in E} f(v, s).$$

A *cut* is a subset $R \subseteq E$, such that removing R disconnects any directed path from the source to the sink. The *capacity of a cut* R is

$$c(R) = \sum_{e\in R} c(e).$$

The *residual capacity* $r(e)$ of an edge e along a path P from z to s is defined as

$$r(e) := \begin{cases} c(e) - f(e) & e \text{ is directed along } P, \\ f(e) & \textbf{otherwise.} \end{cases}$$

A path P is called *augmenting* if all of its edges have positive residual capacity.

Algorithm 1 Ford–Fulkerson(G,s,t,c)

$f \leftarrow$ zero-flow ($f(u, v) = 0\ \forall (u, v) \in E$)
while there exists an augmenting path P from s to t **do**
    $\varepsilon_P \leftarrow \min_{e\in E(P)} r_{f,P}(e)$
    Increase the flow f along P by ε_P
    for all edges e in P **do**
        if e is oriented along the path P **then**
            $f(e) \leftarrow f(e) + \varepsilon_P$
        else
            $f(e) \leftarrow f(e) - \varepsilon_P$
        end if
    end for
end while
return f

Theorem (Max-Flow Min-Cut Theorem). For every network, the maximum value of the flow equals the minimum capacity of a cut:

$$\max_f w(f) = \min_R c(R).$$