Tutorial Sheet 5 - 11.11.2024

Exercise 1 (Error-Correcting Code). Consider the set B_n of binary strings of length n. For $x, y \in B_n$, let $d(x, y)$ denote the number of positions at which the strings differ. For a strong error-correcting code, it is desirable to find a set $M \subseteq B_n$ such that every two strings in M differ in at least 3 positions, i.e.,

$$(\forall x, y \in M : d(x, y) \geq 3)$$

(then we can "correct 1 error"). We construct one such code as follows:

(a) Using Fano's plane, construct the set $F \subseteq B_7$ of seven strings of length 7, where every two strings differ in exactly 4 positions, i.e., $(\forall x, y \in F : d(x, y) = 4)$.

(b) For a given string x, let $\overline{x}$ denote its complement, i.e., a string that has the opposite symbol at each position. Determine $d(x, \overline{x})$ and $d(\overline{x}, y)$ for $x \neq y \in F$.

(c) Find a set $M \subseteq B_7$ consisting of 16 strings such that every two strings differ in at least 3 positions.

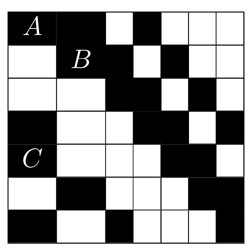
Exercise 2 (Triples of Nine). Given a set D of 3-element subsets of the 9-element set $D = \{1, 2, \ldots, 9\}$, each subset having at most 1 element in common with any other subset, show that:

(a) Every element $x \in D$ belongs to at most four subsets.

(b) The number of subsets can be at most 12.

(c) * Find an example of 12 such subsets.

Exercise 3* (Patches). The cells of a 7×7 grid are colored black and white as shown in the picture. Suppose that two black cells belong to the same "patch" if it is possible to get from one to the other by a sequence of steps, each step moving between two adjacent black cells (sharing an edge). (e.g. cells A and B belong to the same patch, but A and C do not). In one move, Alice can swap any two rows or any two columns of the grid. Show that the black cells will still form exactly 9 patches after any finite number of such moves.



Exercise 4 (Incident graph Graph). Let $(X, \mathcal{P})$ be a finite projective plane of order q . Let's construct a bipartite graph $G = G((X, \mathcal{P}))$ with the classes of bi-partition X and $\mathcal{P}$ so that a point x and a line $p \in \mathcal{P}$ are joined by an edge if and only if x belongs to p.

- Determine the girth g of the graph G . The girth of a graph that contains at least one cycle is the smallest length of such cycle.
- * Determine the number of cycles in G of length g.
- Let H be a bipartite $(n + 1)$-regular graph for$n \geq 2$, without cycles of size 4 and such that between each two vertices there is a path of length at most 3.

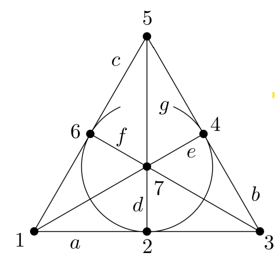
  Prove that H is isomorphic to $G = G((X, \mathcal{P}))$ for a projective plane of order n. .
- What is the minimum number of lines from $\mathcal{P}$(points from X) that we need to select so that every point $x \in X$(every line $P \in \mathcal{P}$) lies on at least one of the selected lines(selected points)?

Finite projective plane:

The pair $(X, \mathcal{P})$ where X is a finite set ("points") and $\mathcal{P} \subseteq 2^X$ is a collection of its subsets ("lines"), such that:

- **(A0)** $(\exists C \subseteq X, |C| = 4)$: $\forall P \in \mathcal{P}$: $|C \cap P| \leq 2$ (There exist 4 points in general position).
- **(A1)** $\forall x \neq y \in X$ $(\exists! P \in \mathcal{P})$: $x, y \in P$ (Every pair of points is contained in exactly one line).
- **(A2)** $\forall P \neq Q \in \mathcal{P}$: $|P \cap Q| = 1$ (Every pair of lines intersects in exactly one point).

Fano plane: The smallest projective plane:



Latin square:

A Latin square is an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.
Two latin squares A, B are **orthogonal Latin squares** if the pair (A_{ij}, B_{ij}) appears only once (and we have all n^2 possible pairs).
A set of Latin squares is **Mutually orthogonal Latin squares**, if each two of them is orthogonal.

Lemma (Latin squares). There is at most $n-1$ Mutually orthogonal Latin squares of size n

Theorem (Latin squares / Projective plane). There are $n-1$ Mutually orthogonal Latin squares of size n **iff** there is a projective plane of size n.

Lemma (Size of a field). Size of a field can only be a power of a prime.

An algebraic construction of projective planes: Define $\mathbb{F}P^2 :=$
$(X =$ lines that pass through (0,0,0) in $\mathbb{F}^3$,
$\mathcal{P} =$ Planes that pass through (0,0,0) (each can represent by exactly one $x \in X$) in $\mathbb{F}^3)$

Theorem (Projective plane). For each field $\mathbb{F}$, $\mathbb{F}P^2$ is a projective plane.