Tutorial Sheet 4 - 4.11.2024

Exercise 0. Construct the incident matrix for a finite projective plane and translate the axioms. Now, explain all your knowledge about finite projective planes in graph language. Also, prove that no two assigned graphs can be a subgraph of one another.

Exercise 1. Seven dwarfs are planning their work week. Each of the 7 days, some 4 of them will go to work (to mine gems), while the remaining 3 will rest and play the card game Mariáš[1]. Is it possible to plan the week so that every pair of dwarfs gets to play Mariáš together at least once?

Exercise 2. Consider a finite projective plane $(X, \mathcal{P})$ of order n.

(a) What is the minimum number of lines from $\mathcal{P}$ that we need to select so that every point $x \in X$ lies on at least one of the selected lines?

(b) What is the minimum number of points from X that we need to select so that every line $P \in \mathcal{P}$ contains at least one of the selected points?

Exercise 3. Give an example of a pair $(X, \mathcal{P})$ that satisfies axioms A1 and A2, but not axiom A0.

Exercise 4. In the card game Dobble, there are 55 cards, each with 8 symbols, and each pair of cards shares exactly one common symbol.

(a) Prove that some symbol appears on at least 8 cards.

(b) Prove that the number of symbols used is greater than the number of cards.

Exercise 5. Prove that it is possible to build several bus routes in a city such that the following conditions hold:

(a) If any single bus line stops functioning, it is still possible to travel between any two stops with at most one transfer.

(b) If any two bus lines stop functioning, the public transportation system remains fully connected.

Exercise 6. The 4-cycle is the graph $C_4 = K_{2,2}$.

(a) For an infinite number of values of n, construct a graph on n vertices with $\Omega(n^{3/2})$ edges, which does not contain a 4-cycle as a subgraph.

(b) (Bonus) Prove that a graph on n vertices without a 4-cycle has at most $O(n^{3/2})$ edges.

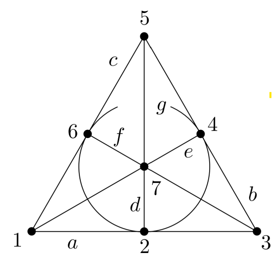
[1] Mariáš or Mariasch a three-player, solo trick-taking game of the king–queen family of ace–ten games, but with a simplified scoring system. It is one of the most popular card games in the Czech Republic and Slovakia, but is also played in Bavaria in Germany as well as in Austria. The Hungarian national card game Ulti is an elaboration of Mariáš.

Finite projective plane:

The pair $(X, \mathcal{P})$ where X is a finite set ("points") and $\mathcal{P} \subseteq 2^X$ is a collection of its subsets ("lines"), such that:

- **(A0)** $(\exists C \subseteq X, |C| = 4)$: $\forall P \in \mathcal{P}$: $|C \cap P| \leq 2$ (There exist 4 points in general position).
- **(A1)** $\forall x \neq y \in X$ $(\exists! P \in \mathcal{P})$: $x, y \in P$ (Every pair of points is contained in exactly one line).
- **(A2)** $\forall P \neq Q \in \mathcal{P}$: $|P \cap Q| = 1$ (Every pair of lines intersects in exactly one point).

Fano plane:



Latin square:

A Latin square is an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.
Two latin squares A, B are **orthogonal Latin squares** if the pair (A_{ij}, B_{ij}) appears only once (and we have all n^2 possible pairs).
A set of Latin squares is **Mutually orthogonal Latin squares**, if each two of them be orthogonal.

Cards arrangement:

Take all aces, kings, queens and jacks from a standard deck of cards, and arrange them in a 4 x 4 grid such that each row and each column contained all four suits as well as one of each face value and also each diagonal contains all four face values and all four suits.

♠ A	♡ K	♢ Q	♣ J
♣ Q	♢ J	♡ A	♠ K
♡ J	♠ Q	♣ K	♢ A
♢ K	♣ A	♠ J	♡ Q

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♠ A	♡ K	♢ Q	♣ J
♢ J	♣ Q	♠ K	♡ A
♣ K	♢ A	♡ J	♠ Q
♡ Q	♠ J	♣ A	♢ K

Thirty-six officers problem:

> "A very curious question, which has exercised for some time the ingenuity of many people, has involved me in the following studies, which seem to open a new field of analysis, in particular the study of combinations. The question revolves around arranging 36 officers to be drawn from 6 different regiments so that they are ranged in a square so that in each line (both horizontal and vertical) there are 6 officers of different ranks and different regiments". - *Leonhard Euler*
> (Euler was unable to solve the problem. He then conjectured that no solution exists if the order of the square is of the form $n = 4k + 2$. This conjecture is false generally, but it's true for $n = 6$.)