Tutorial Sheet 3 - 21.10.2024

Exercise 1. Let's investigate the consequence of the generalized binomial theorem:

(a) What happens when we substitute $x = -y$, $r = -2$? And $x = -y$, $r = -3$?

(b) What is the generating function of the sequence $(1, 2, 3, 4, \ldots, k, \ldots)$?

(c) What is the generating function of the sequence $(0, 1, 2, 2 \cdot 4, 3 \cdot 8, \ldots, k \cdot 2^k, \ldots)$?

(d) What is the generating function of the sequence $(1 \cdot 2, 2 \cdot 3, 3 \cdot 4, \ldots, k(k+1), \ldots)$?

(e) What is the generating function of the sequence $(1, 4, 9, 16, \ldots, k^2, \ldots)$?

(f) (Bonus)What is the generating function of the sequence $(1, 8, 27, 64, \ldots, k^3, \ldots)$?

Exercise 2. Determine the coefficient at the respective powers of x in the following expressions. (Express it in the form $\binom{p}{q}$ for natural numbers p and q without directly calculating it.)

(a) at x^{15} in the expression $(x^2 + x^3 + x^4 + \ldots)^4$,

(b) at x^{28} in the expression $(x + x^3 + x^5 + \ldots)^6$,

(c) at x^5 in the expression $\frac{1}{(1-2x)^2}$.

Exercise 3. Find the formula for the n-th term of the following recursively given sequences:

(a) $a_0 = 2,\ a_1 = 5,\ a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 0$.

(b) $a_0 = 2,\ a_1 = 1,\ a_{n+2} = a_{n+1} + 2a_n$ for $n \geq 0$.

(c) $a_0 = 3,\ a_1 = 5,\ a_{n+2} = 3a_{n+1} - 4a_n$ for $n \geq 0$.

(d) (Bonus) $a_0 = 1,\ a_1 = 9,\ a_{n+2} = 6a_{n+1} - 9a_n$ for $n \geq 0$.

Exercise 4. Express the following quantities using Catalan numbers:

(a) The number of valid bracketings using n symbols "(" and n symbols ")".

(b) The number of paths on a grid from point $[0, 0]$ to point $[n, n]$, which never fall below the diagonal $y = x$.

(c) The number of ways to divide a "staircase" of $1 + 2 + \cdots + n$ squares into n rectangles.

(d) The number of ways to pair up $2n$ points on a circle with n line segments such that no two segments intersect.

(e) The number of triangulations of a convex n-gon.

(f) (Bonus) The number of ways to fill a $2 \times n$ grid with the numbers $1, 2, \ldots, 2n$ such that the numbers increase in each row and column.

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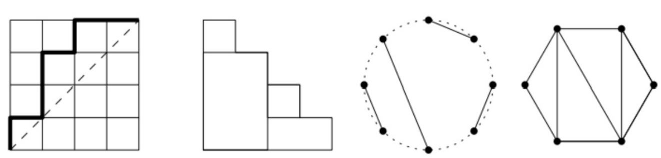

1	3	4
2	5	6

- George Pólya writes in Mathematics and plausible reasoning:

  > A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag.

Generating function	**Sequence**
$a(x)+b(x)$	$(a_0+b_0, a_1+b_1, a_2+b_2, \cdots)$
$\alpha a(x)$	$(\alpha a_0, \alpha a_1, \alpha a_2, \cdots)$
$x^n a(x)$	$\left(\underbrace{0,\cdots,0}_{n}, a_0, a_1, a_2, \cdots\right)$
$\frac{a(x)-a_0-\cdots-a_{n-1}x^{n-1}}{x^n}$	$(a_n, a_{n+1}, a_{n+2}, \cdots)$
$a(\alpha x)$	$(a_0, \alpha a_1, \alpha^2 a_2, \cdots, \alpha^i a_i, \cdots)$
$a\left(x^n\right)$	$\left(\underbrace{a_0, 0, \cdots, 0}_{n}, a_1, 0, \cdots\right)$
$a'(x)$	$(a_1, 2a_2, 3a_3, \cdots, ia_i, \cdots)$
$\int_0^x a(t)dt$	$\left(0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \cdots, \frac{a_i}{i+1}, \cdots\right)$
$a(x)b(x)$	$(c_0, c_1, c_2, \cdots)$, $c_n := \sum_{i=0}^n a_i b_{n-i}$

- **Generalized binomial theorem**
  For an arbitrary **real** number **r**,

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \cdots := \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots$$

- **Linear recurrences (ex: Fibonacci numbers)** For a positive integer k, a *homogeneous linear difference equation of degree* k is an equation of the form

$$y_{n+k} = a_{k-1}y_{n+k-1} + a_{k-2}y_{n+k-2} + \cdots + a_1 y_{n+1} + a_0 y_n,$$

  where $a_{k-1}, \ldots, a_0$ are fixed constants. Often, sequences are defined recursively, by specifying the values of the first k terms, and by a homogeneous linear difference equation of degree k.
  Ex: Fibonacci numbers

$$y_{n+2} = y_{n+1} + y_n \quad y_0 = 1, y_1 = 1$$

  To solve it find the generation function ($\frac{p(x)}{q(x)}$) and then use this fact;

  $\deg p(x) < \deg q(x), \quad q(x) = c\,(x-\alpha_1)^{\beta_1} \ldots (x-\alpha_t)^{\beta_t} \implies \exists A_{i,j} \in \mathbb{C}:$

$$\frac{p(x)}{q(x)} = \frac{A_{1,1}}{x-\alpha_1} + \cdots + \frac{A_{1,\beta_1}}{(x-\alpha_1)^{\beta_1}} + \cdots + \frac{A_{t,1}}{x-\alpha_t} + \cdots + \frac{A_{t,\beta_t}}{(x-\alpha_t)^{\beta_t}}.$$

- **Catalan numbers** The sequence $(1, 1, 2, 5, 14, 42, \ldots)$ of Catalan numbers is defined as

$$c_0 = c_1 = 1 \quad \text{and} \quad c_n = \sum_{i=0}^{n-1} c_i \cdot c_{n-1-i} \quad \text{for} \quad n \geq 2.$$

  It can be proven that

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

  It can also be shown that c_n is the number of rooted binary trees with n internal vertices (and thus $n+1$ leaves).

Tutorial Sheet 3+ - ~~28.10.2024~~

Exercise 1. (Bonus) Determine:

$$\sum_{a,b,c\in N, a+b+c=20} a\cdot b\cdot c$$

Exercise 2. (Sicherman dice) When we roll two regular dice and add the numbers, some sums will be more common than others. Find a way to number the faces of the two dice with appropriate natural numbers differently than usual, so that the two new dice give every possible sum on average as often as the two regular dice.

Exercise 3. Prove Partitions into odd and distinct parts **without** using generating functions.
(Hint: Find a bijective map between these two.)

Exercise 4. [MN-12.6-1] Consider a random walk where we start at the number 0 and in each step we move from i to $i+1$ or to $i-1$ with equal probability.
(a) *Prove that we eventually return to 0 with probability 1.
(b) Prove that each number k is visited at least once with probability 1.

Exercise 5. [MN-12.5-1] Using generating functions as we did above, calculate
(a) the average number of 6 s obtained by rolling a die n times,
(b) *the average value of the expression $(X-6)^2$, where X is the number of times a die must be rolled until the first 6 appears (this is a certain measure of the "typical deviation" from the average number of rounds; in probability theory, it is called the variance).
(c) Let X be some random variable attaining values $0,1,2,\ldots$, where i is attained with probability q_i, and let $q(x)$ be the generating function of the q_i. Express the variance of X, i.e. the quantity $\mathrm{Var}[X] = \mathbf{E}\left[(X-\mathbf{E}[X])^2\right]$, using the derivatives of $q(x)$ at suitable points.