Tutorial Sheet 2 - 14.10.2024

Exercise 1. By using generating functions compute,
(a)$\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}\binom{n}{n-i}$,
(b)$\sum_{k=0}^{n} k \cdot 2^{k}$,
(c)$\sum_{k=0}^{n} k\binom{n}{k}$
(d)$\sum_{k=0}^{n} k^{2}\binom{n}{k}$.

Exercise 2. [MN-12.2] A box contains 30 red, 40 blue, and 50 white balls; balls of the same color are indistinguishable. How many ways are there of selecting a collection of 70 balls from the box?

Exercise 3. [MN-12.2] Find sequences for these generating functions:
(a) $(1+x)\left(1+x^{2}\right)\left(1+x^{4}\right)\ldots\left(1+x^{2^{k}}\right)$
(b) x^{5} in $(1-2x)^{-2}$
(c) $\frac{1}{\sqrt{1-2x}}$
(d) x^{4} in $\left(x^{2}-5x+6\right)^{-1}$

Exercise 4. [MN-12.2-3] Find generating functions for the following sequences (express them a closed form, without infinite series!):
(a) $0,0,0,0,-6,6,-6,6,-6,\ldots$
(b) $1,0,1,0,1,0,\ldots$
(c) $1,2,1,4,1,8,\ldots$
(d) $1,1,0,1,1,0,1,1,0,\ldots$

Exercise 5. [MN-12.2-6] Let a_n be the number of ordered r-tuples $(i_1,\ldots,i_r)$ of nonnegative integers with $i_1+i_2+\cdots+i_r=n$; here r is some fixed natural number.
(a) Find the generating function of the sequence $(a_0,a_1,a_2,\ldots)$.
(b) Find a formula for a_n. (This has been solved by a different method in the first tutorial.)

Exercise 6. [MN-12.3-1]* Determine the number of n-term sequences of 0s and 1s containing no two consecutive 0s.

Exercise 7. [MN-12.3-9*] m is an arbitrary integer, Show that the number $\frac{1}{2}\left[(1+\sqrt{m})^{n}+(1-\sqrt{m})^{n}\right]$ is an integer for all $n \geq 1$.

Exercise 8. (Bonus) Determine:

$$\sum_{a,b,c\in N, a+b+c=20} a \cdot b \cdot c$$

Notes:

Note 1. Dictionary for translating between sequences and their generating functions:

Generating function	**Sequence**
$a(x)+b(x)$	$(a_0+b_0, a_1+b_1, a_2+b_2, \cdots)$
$\alpha a(x)$	$(\alpha a_0, \alpha a_1, \alpha a_2, \cdots)$
$x^n a(x)$	$\left(\underbrace{0, \cdots, 0}_{n}, a_0, a_1, a_2, \cdots\right)$
$\frac{a(x)-a_0-\cdots-a_{n-1}x^{n-1}}{x^n}$	$(a_n, a_{n+1}, a_{n+2}, \cdots)$
$a(\alpha x)$	$(a_0, \alpha a_1, \alpha^2 a_2, \cdots, \alpha^i a_i, \cdots)$
$a\left(x^n\right)$	$\left(\underbrace{a_0, 0, \cdots, 0}_{n}, a_1, 0, \cdots\right)$
$a'(x)$	$(a_1, 2a_2, 3a_3, \cdots, ia_i, \cdots)$
$\int_0^x a(t)dt$	$\left(0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \cdots, \frac{a_i}{i+1}, \cdots\right)$
$a(x)b(x)$	$(c_0, c_1, c_2, \cdots)$, $c_n := \sum_{i=0}^n a_i b_{n-i}$

Note 2. Suppose $p(x)$ and $q(x)$ are polynomials with complex coefficients such that $\deg p(x) < \deg q(x)$. Next, suppose that $q(x)$ can be factored as

$$q(x) = c\left(x-\alpha_1\right)^{\beta_1} \ldots \left(x-\alpha_t\right)^{\beta_t}$$

where c is a non-zero complex number, $\alpha_1, \ldots, \alpha_t$ are pairwise distinct complex numbers, and $\beta_1, \ldots, \beta_t$ are positive integers. 3 In this case [7] there exist complex numbers $A_{1,1}, \ldots, A_{1,\beta_1}, \ldots, A_{t,1}, \ldots, A_{t,\beta_t}$ such that

$$\frac{p(x)}{q(x)} = \frac{A_{1,1}}{x-\alpha_1} + \cdots + \frac{A_{1,\beta_1}}{\left(x-\alpha_1\right)^{\beta_1}} + \cdots + \frac{A_{t,1}}{x-\alpha_t} + \cdots + \frac{A_{t,\beta_t}}{\left(x-\alpha_t\right)^{\beta_t}}.$$

Example:

$$\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}.$$

Note 3. Taylor series formula:

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \cdots$$

So we have this Taylor expansion and this definition:

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \cdots := \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots$$