Suolotions for Tutorial Sheet 2 - 28.10.2024

Exercise 1. (a)

$$(1-x)^n(1+x)^n$$

(b)

$$(\frac{1-x^{n+1}}{1-x}) = f(x) = 1+x+\cdots+x^n \implies x\times\frac{(1-x^{n+1})-(1-x)(n+1)x^n}{(1-x)^2} = xf'(x) = \sum_{k=1}^{n} kx^k$$

$$\implies 2\frac{(1-2^{n+1})-(1-2)(n+1)2^n}{(1-2)^2} = (n-1)2^{n+1}+2 = 2f'(2) = \sum_{k=1}^{n} k2^k$$

(c)

$$(1+x)^n = f(x) = \sum_{k=0}^{n}\binom{n}{k}x^k \implies xn(1+x)^{n-1} = xf'(x) = \sum_{k=0}^{n} k\binom{n}{k}x^k \implies n2^{n-1} = 1f'(1) = \sum_{k=0}^{n} k\binom{n}{k}$$

(d)

$$\textbf{(c)}, g(x) := xf'(x) = xn(1+x)^{n-1} \implies x^2n(n-1)(1+x)^{n-2} + xn(1+x)^{n-1} = xg'(x) = \sum_{k=0}^{n} k^2\binom{n}{k}x^k$$

$$\implies n(n-1)2^{n-2} + n2^{n-1} = 1g'(1) = \sum_{k=0}^{n} k^2\binom{n}{k}$$

Exercise 2. Coefficient x^{70} in

$$f(x) := (1+x+\cdots+x^{30})(1+x+\cdots+x^{40})(1+x+\cdots+x^{50}) = \frac{1-x^{31}}{1-x}\frac{1-x^{41}}{1-x}\frac{1-x^{51}}{1-x} = \frac{(1-x^{31})(1-x^{41})(1-x^{51})}{(1-x)^3}$$

$$(*)\ \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty}\binom{n+k-1}{n-1}x^k \implies \frac{1}{(1-x)^3} = \sum_{k=0}^{\infty}\binom{k+2}{2}x^k$$

$$\implies f(x) = (1-x^{31}-x^{41}-x^{51}+x^{72}+x^{82}+x^{92}-x^{123})\sum_{k=0}^{\infty}\binom{k+2}{2}x^k$$

So the coefficient is

$$\binom{72}{2}-\binom{41}{2}-\binom{31}{2}-\binom{21}{2}$$

Exercise 3. (a)

$$\frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^8}{1-x^4}\cdots\frac{1-x^{2^{k+1}}}{1-x^{2^k}} = \frac{1-x^{2^{k+1}}}{1-x} = \sum_{n=1}^{2^{k+1}-1} x^n$$

(b)

$$* \implies 2^5\binom{2+5-1}{2-1} = 2^5\times 6$$

(c)

$$* \implies \sum_{k=0}^{\infty}\binom{\frac{1}{2}+k-1}{k}2^kx^k = \sum_{k=0}^{\infty}\frac{(k-\frac{1}{2})(k-\frac{1}{2}-1)\dots(k-\frac{1}{2}-(k-1))}{k!}2^kx^k = \sum_{k=0}^{\infty}\frac{(2k-1)(2k-3)\dots(1)}{k!}x^k$$

$$\overset{\times\frac{2^k k!}{2^k k!}}{\Longrightarrow} = \sum_{k=0}^{\infty} \frac{(2k-1)!}{2^k k! k!} x^k = \sum_{k=0}^{\infty} \frac{1}{2^{k+1} k} \binom{2k}{k} x^k$$

(d) x^4 in

$$f(x) = \left(x^2 - 5x + 6\right)^{-1} = ((x-2)(x-3))^{-1} = \frac{-1}{x-2} + \frac{1}{x-3} = \frac{1}{2}\frac{1}{1-\frac{1}{2}x} - \frac{1}{3}\frac{1}{1-\frac{1}{3}x}$$

is equal to

$$\frac{1}{2^5} - \frac{1}{3^5}$$

Exercise 4. **(a)** $0, 0, 0, 0, -6, 6, -6, 6, -6, \ldots$

$$x^4 \frac{6}{1+x}$$

(b) $1, 0, 1, 0, 1, 0, \ldots$

$$\frac{1}{1-x^2}$$

(c) $1, 2, 1, 4, 1, 8, \ldots$

$$\frac{2x}{1-2(x^2)} + \frac{1}{1-x^2}$$

(d) $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

$$\frac{1}{1-x} - \frac{x^2}{1-x^3}$$

Exercise 5. Coefficient x^n in

$$(\frac{1}{1-x})^r$$

Exercise 6. Assume there are F_n possible sequences.

Consider the last term in the sequence. If it is 1, then there are F_{n-1} possible sequences. If it is 0, then the preceding term must be 1, giving

Exercise 7. Coefficient x^i in

$$f(x) := \frac{1}{2}\left[(1+\sqrt{m}x)^n + (1-\sqrt{m}x)^n\right]$$

for odds are 0, for evens is $m^{i/2}\binom{n}{i}$. So, f(1) is an integer.