Tutorial Sheet 10 - 16.12.2024

Exercise 1. The cells of an infinite table are colored with 3 colors. Prove that:

1. There exist ten cells in the same row, all having the same color.
2. There exist two rows and two columns such that the four cells at their intersections have the same color.
3. There exist 10 rows and 10 columns such that the 100 cells at their intersections have the same color.

Exercise 2. Prove that for every sufficiently large natural number, it is possible to delete some of its initial and final digits (not all) so that the resulting number is a multiple of 2027.

Exercise 3(Erdős-Szekeres). Prove that every sequence of n^2+1 distinct numbers contains a monotonic subsequence of length $n+1$.

Exercise 4. Given a graph with n vertices and m edges, where each edge has a distinct weight, prove that there exists a path through at least $\lceil 2m/n \rceil$ edges where the weights increase consecutively.

Exercise 5. For $N \in \mathbb{N}$, let K_N be the complete graph with vertices $V = \{1, 2, \ldots, N\}$. Determine which of the following statements are true:

1. For every sufficiently large $N \in \mathbb{N}$, whenever the edges of K_N are colored with two colors, there exists a monochromatic complete subgraph on 10 vertices.
2. For every sufficiently large $N \in \mathbb{N}$, whenever the edges of K_N are colored with two colors, there exists a monochromatic subgraph $K_{10,10}$ on 20 vertices.
3. For every sufficiently large $N \in \mathbb{N}$, whenever the edges of K_N are colored with two colors, there exists a monochromatic complete subgraph on 10 vertices containing vertex 1.
4. For every sufficiently large $N \in \mathbb{N}$, whenever the edges of K_N are colored with two colors, there exists a monochromatic complete subgraph on 10 vertices, all of whose vertices are powers of 2.
5. For every sufficiently large $N \in \mathbb{N}$, whenever the edges and loops of K_N are colored with two colors, there exists a monochromatic complete subgraph on 10 vertices (including loops).

Exercise 6. Prove that $R_2(3,3,3) \leq 17$.

Exercise 7. The edges of the graph K_9 are colored red and blue. Prove that there exists either a blue triangle C_3 or a red quadrilateral C_4.

Exercise 8 . Prove that $R_2(n,m) \leq R_2(n,m-1) + R_2(n-1,m)$ for every $n, m \in \mathbb{N}$.

Pigeonhole principle: Whenever we distribute n balls into k boxes, there will be at least $\lceil n/k \rceil$ balls in one box.

We color the edges of the complete graph K_N with k colors and look for monochromatic cliques. For $n_1, \ldots, n_k \in \mathbb{N}$, define the Ramsey number $R_2(n_1, \ldots, n_k)$ as the smallest $N \in \mathbb{N}$ such that for any coloring of the edges of K_N with k colors, there exists a color $i \in [k]$ such that the subgraph K_{n_i} is monochromatic in color i.

Theorem (Ramsey): For every $n_1, \ldots, n_k \in \mathbb{N}$, the number $R_2(n_1, \ldots, n_k)$ is finite.

$\mathbf{R_2(3,3) = 6}$, $R_2(4,4) = 18$, $R_2(5,5) = ?$