Tutorial Sheet 1 - 7.10.2024

Exercise 0. [MN-3.5-10]Prove the following upper bound for the factorial function (which is already quite close to Stirling's formula):$n! \leq \mathrm{e}\sqrt{n}(n/\mathrm{e})^n$.

Exercise 1. [MN-3.1-1] Let $X = \{x_1, x_2, \ldots, x_n\}$ be an n-element set. Describe how each subset of X can be encoded by an n-letter word consisting of the letters a and b. Infer that the number of subsets of X is 2^n.

Exercise 2. [MN-3.1-2] Determine the number of ordered pairs (A, B), where $A \subseteq B \subseteq \{1, 2, \ldots, n\}$.

Exercise 3. [MN-3.1-6] Show that a natural number $n \geq 1$ has an odd number of divisors (including 1 and itself) if and only if $\sqrt{n}$ is an integer.

Exercise 4. [MN-3.2-1] How many permutations of $\{1, 2, \ldots, n\}$ have a single cycle?

Exercise 5. $p = (483529617)$,What is the minimum number k such that permutation p^k be equal to identity function?

Exercise 6. [MN-3.3-7] How many functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there that are monotone; that is, for $i < j$ we have $f(i) \leq f(j)$?

Exercise 7. [MN-3.3-3] Prove the formula

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

Exercise 8. [MN-3.3-8] How many terms are there in the sum on the right-hand side of the formula for $(x_1 + \cdots + x_m)^n$ in the multinomial theorem?

Exercise 9. For very large n, order the following expressions by size:

$$1000n \quad \frac{1}{2}n(n+1) \quad 1.1^n \quad n\sqrt{n} \quad n \log n$$

Exercise 10. For very large n, order the following expressions by size:

$$\binom{2n}{n} \quad \binom{2n}{5} \quad n! \quad n^n \quad (\sqrt{n})^n \quad n\sqrt{n} \quad n^5$$

Exercise 11. [MN-3.5-3] prove that For every real number x,

$$1 + x \leq \mathrm{e}^x$$

Exercise 12. [MN-3.5-13] Let H_n be $H_n = \sum_{i=1}^{n} \frac{1}{i}$, prove the inequalities $\ln n < H_n \leq \ln n + 1$ by induction on n.

Notes:

Note 0. For every $m, n \geq 1$ and for every $k, 1 \leq k \leq n$; we have,

$$\mathrm{e}\left(\frac{n}{\mathrm{e}}\right)^n \leq n! \leq \mathrm{e}n\left(\frac{n}{\mathrm{e}}\right)^n, \quad \left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{\mathrm{e}n}{k}\right)^k, \quad \frac{2^{2m}}{2\sqrt{m}} \leq \binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{2m}}.$$

Note 1. For a (counting) function $f(n)$ when n gets larger and larger we care less about the difference between n^{10} and $(2)n^{10} + n^8$ and more about the difference between n^8 and n^{10} and much more about the difference between each of them and 2^n or $log(n)$. We have the following notations.

Notation	**Definition**
$f(n) = O(g(n))$	$\exists n_0 \in \mathbb{N}, C \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, if $n \geq n_0$ then $\lvert f(n)\rvert \leq Cg(n)$
$f(n) = o(g(n))$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
$f(n) = \Omega(g(n))$	$g(n) = O(f(n))$
$f(n) = \Theta(g(n))$	$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
$f(n) \sim g(n)$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

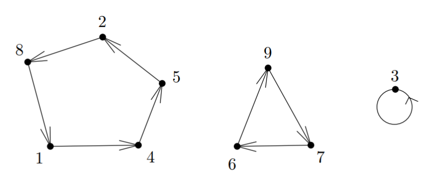
Note 3. Probably lots of you forget Stirling's formula many times,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Note 4. Let $\pi(n)$ denote the number of primes not exceeding the number n. Then[1]

$$\pi(n) \sim \frac{n}{\ln n}.$$

Note on Exercise 5. This is the Cycle decomposition of permutation $p = (483529617)$:



Riemann zeta function is a function of a complex variable defined as,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

As we saw in the random walk on a line $\zeta(1) = \infty$; so we have the condition $\mathrm{Re}(s) > 1$.
Do you know what is $\sum_{n=0}^{\infty} \frac{1}{n^2}$? $\zeta(2) = \frac{\pi^2}{6}$.

[1]See **[MN-3.6-2]**.Don't confuse this function with number $\pi = 3.141592653589\ldots$.