HW5

November 12, 2024

Name: ____

Finite projective plane:
The pair $(X, \mathcal{P})$ where X is a finite set ("points") and $\mathcal{P} \subseteq 2^X$ is a collection of its subsets ("lines"), such that:

- **(A0)** $(\exists C \subseteq X, |C| = 4)$: $\forall P \in \mathcal{P}$: $|C \cap P| \leq 2$ (There exist 4 points in general position).
- **(A1)** $\forall x \neq y \in X$ $(\exists! P \in \mathcal{P})$: $x, y \in P$ (Every pair of points is contained in exactly one line).
- **(A2)** $\forall P \neq Q \in \mathcal{P}$: $|P \cap Q| = 1$ (Every pair of lines intersects in exactly one point).

1. Prove that there is a labeling of points(from 1 to n^2+n+1) and a labeling of lines(from 1 to n^2+n+1), such that the line i is incident with the point i (for $i = 1$ to n^2+n+1)[Hint: you can use the incidence graph and Hall's theorem,]

2. (Error-Correcting Code) Consider the set B_n of binary strings of length n. For $x, y \in B_n$, let $d(x, y)$ denote the number of positions at which the strings differ. For a strong error-correcting code, it is desirable to find a set $M \subseteq B_n$ such that every two strings in M differ in at least 5 positions, i.e.,

$$(\forall x, y \in M : d(x, y) \geq 5)$$

(then we can "correct 2 errors"). We construct one such code as follows:

   (a) Using the Projective plane of order 4, construct the set $F \subseteq B_{21}$ of 21 strings of length 21, where every two strings differ in exactly 8 positions, i.e., $(\forall x, y \in F : d(x, y) = 8)$.

   (b) For a given string x, let $\overline{x}$ denote its complement, i.e., a string that has the opposite symbol at each position. Determine $d(x, \overline{x})$ and $d(\overline{x}, y)$ for $x \neq y \in F$.

   (c) Find a set $M \subseteq B_{21}$ consisting of 44 strings such that every two strings differ in at least 5 positions. (It's not an efficient way to make 2-errors correcting code.)