HW3

November 11, 2024

Name: ____

1. Find the coefficient of x^n in the power series for

$$f(x) = \frac{1}{(1-x^2)^2},$$

2. Consider a product of 4 numbers, abcd. It can be "parenthesized" in 5 ways:

$$((ab)c)d, (a(bc))d, (ab)(cd), a((bc)d) \text{ , and } a(b(cd)).$$

Prove that the number of such parenthesizings of a product of n numbers is the Catalan number b_{n-1}.

Bonus This exercise is related to the theory of formal languages. We give the necessary definitions but to appreciate the context, the reader should consult a textbook on automata and formal languages.

Let Σ be some fixed finite alphabet (such as $\Sigma = \{a, b, c\}$). A word over Σ is a finite sequence of letters (such as *babbaacccba*). The empty word having no letters is denoted by ϵ. A language over Σ is a set of words over Σ. If u and v are words then uv denotes the *concatenation* of u and v, i.e. we write first u and then v. The *generating function* of a language L is the generating function of the sequence $n_0, n_1, n_2, \ldots$, where n_i is the number of words of length i in L.

Let us say that a language L is *very regular* if it can be obtained by finitely many applications of the following rules:

- The languages $\emptyset$ and $\{\epsilon\}$ are very regular, and the language $\{\ell\}$ is very regular for every letter $\ell \in \Sigma$.

- If L_1, L_2 are very regular languages, then also the language $L_1.L_2$ is very regular, where $L_1.L_2 = \{uv : u \in L_1, v \in L_2\}$.

- If L is a very regular language, then also L^* is very regular, where $L^* = \{\epsilon\} \cup L \cup L.L \cup L.L.L \cup \ldots$.

- If L_1, L_2 are very regular languages with $L_1 \cap L_2 = \emptyset$, then also $L_1 \cup L_2$ is very regular.

(a) Show that the following languages over $\Sigma = \{a, b\}$ are all very regular: the language consisting of all words of odd length, the language consisting of all words beginning with *aab* and having an even number of letters a, and *the language consisting of all words having no two consecutive letters a.*

(b) * Show that the generating function of any very regular language is a rational function (a polynomial divided by a polynomial), and describe how to calculate it.

(c) ** Show that if L_1, L_2 are very regular languages (not necessarily disjoint ones) then also $L_1 \cup L_2$ is very regular. (Therefore, regular languages have rational generating functions.)