HW1

October 14, 2024

Name: ____

1. (Bonus) (Tschebyshev estimate of $\pi(n)$)

   Let $\pi(n)$ denote the number of primes not exceeding the number n. Then

   $$\pi(n) \sim \frac{n}{\ln n}.$$

   (a) Show that the product of all primes p with $m < p \leq 2m$ is at most $\binom{2m}{m}$.
   (b)* Using (a), prove the estimate $\pi(n) = O(n/\ln n)$.
   (c)* Let p be a prime, and let m, k be natural numbers. Prove that if p^k divides $\binom{2m}{m}$ then $p^k \leq 2m$.
   (d) Using (c), prove $\pi(n) = \Omega(n/\ln n)$.

2. For very large n, order the following expressions by size and explain it:
   (a)

   $$\binom{2n}{n} \quad \binom{2n}{5} \quad n! \quad n^n \quad (\sqrt{n})^n \quad n\sqrt{n} \quad n^5$$

   We already know the answer! but why?
   (b)

   $$n\ln n, \quad (\ln\ln n)^{\ln n}, (\ln n)^{\ln\ln n}, \quad n{\cdot}e^{\sqrt{\ln n}}, \quad (\ln n)^{\ln n}, \quad n{\cdot}2^{\ln\ln n} \quad n^{1+1/(\ln\ln n)}, n^{1+1/\ln n}, n^2$$

3. Let $x_1, x_2, \ldots, x_n$ be positive reals. Their arithmetic mean equals $(x_1 + x_2 + \cdots + x_n)/n$, and their geometric mean is defined as $\sqrt[n]{x_1 x_2 \ldots x_n}$. Let $AG(n)$ denote the statement "for any n-tuple of positive reals $x_1, x_2, \ldots, x_n$, the geometric mean is less than or equal to the arithmetic mean". Prove the validity of $AG(n)$ for every n by the following strange induction:
   (a) Prove that $AG(n)$ implies $AG(2n)$, for each n.
   (b) *Prove that $AG(n)$ implies $AG(n-1)$, for each $n > 1$.
   (c) Explain why proving (a) and (b) is enough to prove the validity of $AG(n)$ for all n