Covering lattice points by subspaces and counting point-hyperplane incidences

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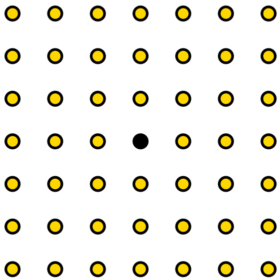
- Affine subspaces: For $n \in \mathbb{N}$, what is the minimum number of lines needed to cover the $n \times n$ integer lattice?
- Linear subspaces: What if all the lines have to contain the origin?



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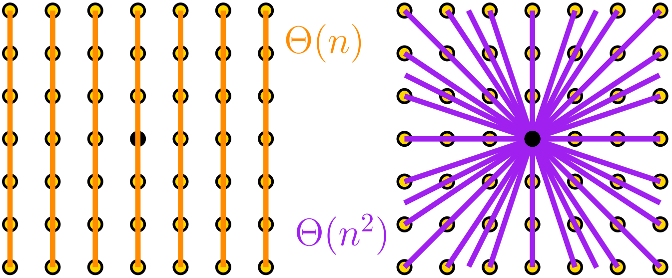


Problem 1 (Brass, Moser, Pach, 2005)

Let k be an integer with $1 \leq k \leq d-1$. What is the minimum number of k-dimensional linear subspaces needed to cover the d-dimensional $n \times \cdots \times n$ lattice?

- For affine subspaces the answer is n^{d-k}.
- For $k = 1$ the answer is $\Theta(n^d)$.
- Bárány, Harcos, Pach, Tardos (2001): For $k = d-1$, i.e. for hyperplanes containing the origin, the answer is $\Theta(n^{d/(d-1)})$.

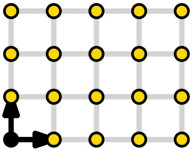
Main Result

For k with $1 \leq k \leq d-1$ and $n \in \mathbb{N}$, the $n \times \cdots \times n$ lattice can be covered with $O(n^{d(d-k)/(d-1)})$ k-dimensional linear subspaces and for every $\varepsilon > 0$ we need at least $\Omega(n^{d(d-k)/(d-1)-\varepsilon})$ k-dimensional linear subspaces to cover it.

General lattices and symmetric convex bodies

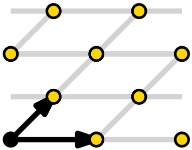
- $\mathcal{L}^d$: d-dimensional lattices ... For linearly independent vectors $b_1, \dots, b_d \in \mathbb{R}^d$, the d-dimensional lattice $\Lambda \in \mathcal{L}^d$ with basis $\{b_1, \dots, b_d\}$ is the set

$$\Lambda = \{a_1 b_1 + \cdots + a_d b_d \colon a_1, \dots, a_d \in \mathbb{Z}\}.$$



$b_1 = (1, 0)$
$b_2 = (0, 1)$

$b'_1 = (2, 0)$
$b'_2 = (1, 1)$



- $\mathcal{K}^d$: Convex bodies symmetric around 0.

Generalized problem 1

For $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$, what is the minimum number of k-dimensional linear subspaces needed to cover $\Lambda \cap K$?

Simplifications:

- Replacing K with its John's ellipsoid affects only constants in our estimates $\Rightarrow$ Only ellipsoids from $\mathcal{K}^d$.
- Linear transformations do not affect the covering $\Rightarrow$ We can choose one of:
  - K is a ball from $\mathcal{K}^d$ and $\Lambda \in \mathcal{L}^d$
  - K is an ellipsoid from $\mathcal{K}^d$ and Λ is an integer lattice

How to cover integer points in an ellipsoid K with affine subspaces?

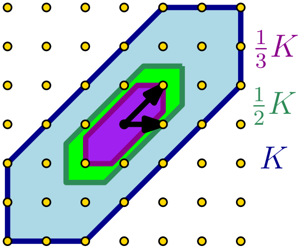
- Easy: If K is the $n_1 \times n_2 \times \cdots \times n_d$ axis-parallel box, where $n_1 \geq n_2 \geq \cdots \geq n_d$, we use $n_{k+1} n_{k+2} \cdots n_d$ k-dimensional affine subspaces.
- For general ellipsoid K: Find a basis of the integer lattice such that, after a linear transformation sending the basis vectors to unit vectors and K to K', K' can be enclosed in a small axis-parallel box.

Successive minima

- For $i = 1, \ldots, d$, the ith successive minimum of Λ and K is

$$\lambda_i = \lambda_i(\Lambda, K) = \inf\{\lambda \in \mathbb{R}\colon \dim(\Lambda \cap (\lambda \cdot K)) \geq i\}$$

  Its corresponding vector is some $v_i \in \Lambda \cap (\lambda_i \cdot K)$ that is linearly independent with $\{v_1, v_2, \ldots, v_{i-1}\}$.



$$\lambda_1(\mathbb{Z}^2, K) = 1/3$$
$$v_1(\mathbb{Z}^2, K) = (1, 1)$$

$$\lambda_2(\mathbb{Z}^2, K') = 1/2$$
$$v_2(\mathbb{Z}^2, K) = (1, 0)$$

- The vectors v_i do not always form a basis of Λ, but we can use:

First finiteness theorem (Mahler 1938, Weyl 1942)

Let d be a positive integer. For every $\Lambda \in \mathcal{L}^d$ and every $K \in \mathcal{K}^d$, there is a basis $\{b_1, \dots, b_d\}$ of Λ with $b_i \in (3/2)^{i-1}\lambda_i(\Lambda, K) \cdot K$ for every $i \in [d]$.

- We then show that there are constants c_i such that all points from $\Lambda \cap K$ can be written as $\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_d b_d$, where α_i are integers satisfying $|\alpha_i| \leq c_i/\lambda_i$.
- Important tool:

Minkowski's second theorem, 1910

Let d be a positive integer. For every $\Lambda \in \mathcal{L}^d$ and every $K \in \mathcal{K}^d$, we have

$$\mathrm{vol}(K) \leq \frac{2^d \det(\Lambda)}{\lambda_1(\Lambda, K) \cdots \lambda_d(\Lambda, K)} \leq d! \cdot \mathrm{vol}(K).$$

Theorem 3
For k with $1 \leq k \leq d-1$, $\Lambda \in \mathcal{L}^d$, and $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, the set $\Lambda \cap K$ can be covered with

$$O((\lambda_{k+1} \cdots \lambda_d)^{-1})$$

k-dimensional affine subspaces and this is tight.

Theorem (Bárány, Harcos, Pach, Tardos, 2001)

For $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$ with $\lambda_d \leq 1$, the set $\Lambda \cap K$ can be covered with at most

$$O\left(\min_{1\leq j\leq d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}\right)$$

$(d-1)$-dimensional linear subspaces and this is tight if λ_d is not close to 1.

- For $\Lambda = \mathbb{Z}^d$ and $K = [-n, n]^d$, we have $\lambda_1 = \cdots = \lambda_d = 1/n$ and thus $j = 1$, which gives the $\Theta(n^{d/(d-1)})$ bound.

Our results – covering by linear subspaces

Theorem 1

For k with $1 \le k \le d-1$, $\Lambda \in \mathcal{L}^d$, and $K \in \mathcal{K}^d$ with $\lambda_d \le 1$, we can cover $\Lambda \cap K$ with $O(\alpha^{d-k})$ k-dimensional linear subspaces, where

$$\alpha = \min_{1 \le j \le k} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}.$$

Theorem 2

For k with $1 \le k \le d-1$, $\Lambda \in \mathcal{L}^d$, $K \in \mathcal{K}^d$ with $\lambda_d \le 1$, and $\varepsilon \in (0,1)$, we need at least $\Omega(((1-\lambda_d)\beta)^{d-k-\varepsilon})$ k-dimensional linear subspaces to cover $\Lambda \cap K$, where

$$\beta = \min_{1 \le j \le d-1} (\lambda_j \cdots \lambda_d)^{-1/(d-j)}.$$

- The bounds are not tight. Can the lower bound be improved?

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Covering by linear subspaces – sketches of proofs

Sketch of proof of Theorem 1: Based on the values of successive minima, either

- Cover $\Lambda \cap K$ by hyperplanes and cover the points within each hyperplane using the induction hypothesis.
- Cover $\Lambda \cap K$ by $(k-1)$-dimensional affine subspaces and extend the subspaces to k-dimensional linear subspaces.

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Theorem 2 follows from the following

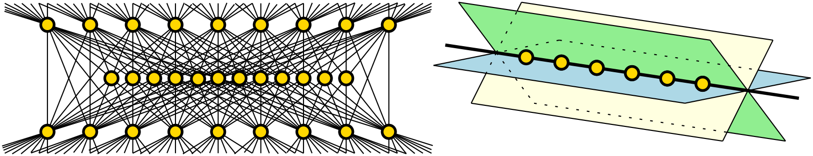
Theorem 3

Given k with $1 \leq k \leq d-1$, $\Lambda \in \mathcal{L}^d$ and $K \in \mathcal{K}^d$. If $\lambda_d \leq 1$, then, for every $\varepsilon \in (0,1)$, there is a positive integer $r = r(d, \varepsilon, k)$ and a set $S \subseteq \Lambda \cap K$ of size at least $\Omega_{d,\varepsilon,k}(((1-\lambda_d)\beta)^{d-k-\varepsilon})$, such that every k-dimensional linear subspace of $\mathbb{R}^d$ contains at most $r-1$ points from S.

- For $k = d-1$, a better bound $\Omega_{d,k}(((1-\lambda_d)\beta)^{d-k})$ was proved by Bárány, Harcos, Pach, Tardos.

Application: bounds for point-hyperplane incidences

- An incidence between an n-point set $P \subseteq \mathbb{R}^d$ and a set of m hyperplanes $\mathcal{H}$ in $\mathbb{R}^d$ is a pair (p, H) such that $p \in P$, $H \in \mathcal{H}$, and $p \in H$.
- What is the maximum number of incidences between P and $\mathcal{H}$ in $\mathbb{R}^d$?



- In the plane, the Szemerédi–Trotter Theorem says that it is at most $O((mn)^{2/3} + m + n)$ for all P and $\mathcal{H}$. Moreover, this is tight.
- For $d \geq 3$ it is trivially at most mn and this is tight!
- To avoid this, we forbid $K_{r,r}$ for some fixed r in the incidence graph.
- Then the maximum number of incidences is at most $O\left((mn)^{1-1/(d+1)} + m + n\right)$ (Chazelle, 1993).

Our results – counting point-hyperplane incidences

Theorem (Brass and Knauer, 2003)

For $d \geq 3$, $\varepsilon > 0$ there is an r such that for all n and m there is a set P of n points in $\mathbb{R}^d$ and a set $\mathcal{H}$ of m hyperplanes in $\mathbb{R}^d$ with no $K_{r,r}$ in the incidence graph and with the number of incidences at least

$$\begin{aligned} &\Omega\left((mn)^{1-2/(d+3)-\varepsilon}\right) && \text{if } d \text{ is odd and } d > 3,\\ &\Omega\left((mn)^{1-2(d+1)/(d+2)^2-\varepsilon}\right) && \text{if } d \text{ is even},\\ &\Omega\left((mn)^{7/10}\right) && \text{if } d = 3.\end{aligned}$$

- For $d \geq 4$, Theorem 3 improves these bounds to

$$\begin{aligned} &\Omega\left((mn)^{1-(2d+3)/((d+2)(d+3))-\varepsilon}\right) && \text{if } d \text{ is odd},\\ &\Omega\left((mn)^{1-(2d^2+d-2)/((d+2)(d^2+2d-2))-\varepsilon}\right) && \text{if } d \text{ is even}.\end{aligned}$$

- The gap in the exponents is of order $\Theta(1/d)$ and the improvement is of order $\Theta(1/d^2)$.

Sketch of the proof:

- Brass and Knauer used a weaker version of Theorem 3. Using Theorem 3 gives the improvement.
- P are the integer points from a ball with no r points on a k-dimensional affine subspace, where $k = \lfloor (d-2)/2 \rfloor$
- N are the integer points from a ball of different radius with no r points on a $(d-k-1)$-dimensional linear subspace.
- $\mathcal{H}$ are hyperplanes with a normal vector in N that contain at least one point from P
- The conditions of no r points on the affine/linear subspace guarantee the absence of $K_{r,r}$.

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Thank you.