Mathematics++

Practicals 3 – Functional analysis

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All the vector spaces (also called linear spaces) are over the field $\mathbb{R}$.

Definition: Let E be a normed linear space. A **closed hyperplane** is every set of the form $H = \{x \in E : f(x) = \alpha\}$ where $f \in E^*$, $f \neq 0$ and $\alpha \in \mathbb{R}$. (This is the same as translations of maximal proper subspaces).

1. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| < |x - y|$ but f is not a contraction.
2. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| < |x - y|$ but f has no fixed point.
3. Show that every subspace of a normed linear space of finite dimension is closed and find a counterexample for a space of infinite dimension.
4. Show that unit ball in a Hilbert space of infinite dimension is not compact.
5. Prove Mazur theorem: Let C be an open convex subset of a normed linear space E and $z \in E \setminus C$. Then there exists a closed hyperplane $H \subset E$ such that $z \in H$ and $H \cap C = \emptyset$.
6. Decide whether following functionals on a normed linear space X are linear and continuous. If so, determine their norm.
   (a) $F : (x_n)_{i \in \mathbb{Z}^+} \mapsto \sum_{i=1}^{\infty} \frac{x_i}{i^2}$, $X = c_0$
   (b) $F : f \mapsto \int_0^1 t f(t)\,\mathrm{d}t$, $X = L^p([0,1])$
   (c) $F : f \mapsto \lim_{n\to\infty} \int_0^1 f(t^n)\,\mathrm{d}t$, $X = \mathcal{C}([0,1])$