Topological methods in combinatorics

Problem set 5 – Non-embeddability, Homology

Submitted: **09. 05. 2024** - Hints: **on individual request (via Owl)** - Deadline: **30. 06. 2024**

Submit solution via the Postal Owl

1. (a) Find a map $f : S^2 \to D_1 * K_4$ that does not identify antipodal points. **[2]**

   (b) Find a map $g : S^5 \to D_1 * K_4 * K_4$ that does not identify antipodal points. **[2]**

   (c) Prove that $K_4 * K_4$ cannot be embedded to $\mathbb{R}^4$. **[1]**

2. Let $f : X \to Y$ be a simplicial map between triangulated spaces. Prove that the associated maps $f_\# : C_n(X) \to C_n(Y)$ satisfy the relation $\partial f_\# = f_\# \partial$. **[2]**

3. Compute the homology groups of $\partial\Delta_d$ from the definition of simplicial homology. You can use that $H_0(\Delta_d) \cong \mathbb{Z}$, and $H_n(\Delta_d) \cong 0$ for $n \geq 1$. **[3]**

4. Let K be a simplicial complex such that $|\mathsf{K}|$ has k path-connected components. Prove that $H_0(\mathsf{K}) \cong \mathbb{Z}^k$. **[2]**

5. If $A \subseteq X$, a retraction $r : X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$. Show that there is no retraction $r : S^1 \times B^2 \to S^1 \times S^1$. **[3]**

6. Find an arbitrary triangulation of the projective plane and a homologically non-trivial 1-cycle c with the property that $2c$ is homologically trivial. That is, find $c \in \operatorname{Ker} \partial_1$ such that $[c] \neq 0$ and $[2c] = 0$. **[3]**