Topological methods in combinatorics

Problem set 4 – The Ham sandwich theorem, joins

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1. The ham sandwich theorem for mass distributions says the following:

   Given mass distributions $\mu_1, \dots, \mu_d$ (finite Borel measures assigning 0 to hyperplanes) on $\mathbb{R}^d$, there is a hyperplane h such that $\mu_i(h^+) = \mu_i(h^-)$ for all $i \in [d]$, where h^+ and h^- are the two open half-spaces defined by h.

   Show the ratio 1 : 1 is the only ratio such that any two compact sets in the plane can be simultaneously partitioned by a line in that ratio. **[1]**

2. The ham sandwich theorem for point sets says the following:

   Given finite sets $A_1, \dots, A_d$ of points in $\mathbb{R}^d$, there is a hyperplane h that contains at most $\left\lfloor \frac{|A_i|}{2} \right\rfloor$ points from each set A_i in each open half-space determined by h.

   Show that if we replace the original definition of bisecting by more natural definition: a hyperplane h bisects $A \subset \mathbb{R}^d$, if $|h^+ \cap A| + \frac{1}{2}|h \cap A| = \frac{1}{2}|A|$, where h^+ is one of the open half-spaces defined by h, then there exist two finite point sets in the plane which cannot be bisected. **[2]**

3. The theorem of Akiyama and Alon says the following:

   Consider sets $A_1, A_2, \dots, A_d$, of n points each, in general position in $\mathbb{R}^d$; imagine that the points of A_1 are red, the points of A_2 blue, etc. (each A_i has its own color). Then the points of the union $A_1 \cup \dots \cup A_d$ can be partitioned into "rainbow" d-tuples (each d-tuple contains one point of each color) with disjoint convex hulls.

   Prove the planar case ($d = 2$) of this theorem by considering a perfect red–blue matching with the minimum possible total length of the edges. In other words, prove that such a matching length does not contain crossing edges. **[1]**

4. Prove that any mass distribution in the plane can be dissected into 6 equal parts by 3 lines passing through a common point. (If you are using a continuity of certain functions, it is enough to justify the continuity informally.) **[4]**

   Definition (Join of simplicial complexes). *Let K and L be simplicial complexes with disjoint vertex sets. The join $\mathsf{K} * \mathsf{L}$ is the simplicial complex with the vertex set $V(\mathsf{K}) \cup V(\mathsf{L})$ and the set of simplicies $\{F \cup G : F \in \mathsf{K}, G \in \mathsf{L}\}$.*

   Definition (Join of topological spaces). *Let X and Y be topological spaces. The join $X * Y$ is the quotient space $X \times Y \times [0,1]/\approx$, where the equivalence relation $\approx$ is given by $(x, y, 0) \approx (x', y, 0)$ for all $x, x' \in X$ and all $y \in Y$, and $(x, y, 1) \approx (x, y', 1)$ for all $x \in X$ and all $y, y' \in Y$.*

   In the next exercise, you may use the following lemma.

Lemma. *Consider two bounded subspaces* X, Y *of* $\mathbb{R}^n$ *such that* $X \subseteq U, Y \subseteq V$*, where* U *and* V *are skew affine subspaces of* R^n *(that is,* $U \cap V = \emptyset$ *and the affine hull of* $U \cup V$*,* $\mathrm{aff}(U \cup V)$*, has dimension* $\dim U + \dim V + 1$*). Then the space*

$$Z = \{tx + (1-t)y : t \in [0,1], x \in X, y \in Y\}$$

is homeomorphic to the join $X * Y$*.*

5. (a) Let U and V be skew affine subspaces in $\mathbb{R}^n$, and let $A \subset U$ and $B \subset V$ be affinely independent sets. Check that $A \cup B$ is affinely independent. **[2]**

   (b) Verify that the union of all segments connecting a point of $\mathrm{conv}(A)$ to a point of $\mathrm{conv}(B)$ is the simplex $\mathrm{conv}(A \cup B)$. **[2]**

   (c) Use the lemma to prove that $|\mathsf{K} * \mathsf{L}| \cong |\mathsf{K}| * |\mathsf{L}|$ for any two finite simplicial complexes K and L (assuming that $|\mathsf{K}|$ and $|\mathsf{L}|$ lie in skew affine subspaces). **[2]**