Topological methods in combinatorics

Problem set 3 – The Borsuk-Ulam Theorem and its applications

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1. Prove directly the 1-dimensional version of Lyusternik-Shnirelman theorem (LS-o). In other words, prove, that for each covering of S^1 by two open sets there is a pair of antipodal points in one of these two sets. **[2]**

2. Let the torus be represented as $T = S^1 \times S^1 \in \mathbb{R}^4$.

   (a) Show that there is a continuous map $f\colon T \to \mathbb{R}^2$ for which there is no $x \in T$ such that $f(x) = f(-x)$. **[1]**

   (b) Show that for each continuous $f\colon T \to \mathbb{R}$ there is an $x \in T$ such that $f(x) = f(-x)$. **[2]**

3. Show that the following statement is equivalent to one of the versions of the Borsuk-Ulam theorem: Whenever S^n is covered by $n+1$ sets $A_1, A_2, \ldots, A_{n+1}$, each A_i open or closed, there is an i such that $A_i \cap (-A_i) \neq \emptyset$. **[2]**

4. Show that the following statement is equivalent to one of the versions of the Borsuk-Ulam theorem: If $f\colon S^n \to S^n$ is antipodal, then every mapping $g\colon S^n \to S^n$ which is homotopic to f is surjective. **[2+2]**

5. Let $\mathrm{SG}(n,k)$ denote *Schrijver graph* whose vertices are stable k-element subsets of $\{1,\ldots,n\}$ [1] and whose two vertices form an edge if and only if the corresponding k-element subsets are disjoint.

   (a) Show that the Schrijver graph $\mathrm{SG}(n,k)$ is not regular in general; that is, its vertices need not all have the same degree. **[2]**

   (b) Show that not all $\mathrm{SG}(n,k)$ are edge-critical (an edge may be removed without decreasing the chromatic number). **[3]**

[1] Recall that $S \subseteq \{1,\ldots,n\}$ is *stable* if it does not contain any two adjacent elements modulo n. That is, if $i \in S$, then $i+1 \notin S$, and if $n \in S$, then $1 \notin S$.