Topological methods in combinatorics

Problem set 2 – Homotopy and simplicial complexes

Published: **14.03.2024** - Hints: **28. 03. 2024** - Deadline: **04. 04. 2024**
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1. (Elementary collapse) Let K be a simplicial complex and let $\sigma \in K$ be contained in a unique maximal face $\tau \in K$, $\sigma \subsetneq \tau$. By an *elementary collapse* we mean an operation that produces a new complex K' from K. The faces of K' are the faces of K except by $\{\mu \in K : \sigma \subseteq \mu \subseteq \tau\}$. Show that K and K' are homotopy equivalent. **[3]**

2. (Barycentric subdivision) Let K be a simplicial complex, show that the geometric realization of K and its barycentric subdivision are homeomorphic, i.e. $|K|$ and $|\operatorname{sd}(K)|$ are homeomorphic. **[4]**

3. (Diameter of barycentric subdivision)

   (a) Prove that the diameter of an arbitrary simplex σ is equal to the distance between some two vertices of σ. **[2]**

   (b) Prove that for every n and $\delta > 0$ there exists k such that if σ^n is any n-dimensional simplex of diameter 1, then all the simplices of $\operatorname{sd}^k(\sigma^n)$, i.e. barycentric subdivision iterated k times, have diameter at most δ. Does k have to depend on n? **[2]**

4. Let $s\colon K \to L$ and $l\colon L \to Z$ be a simplicial maps.

   (a) Show, in detail, that the affine extension $|s|\colon |K| \to |L|$ is a continuous function. **[2]**

   (b) Show that $|l \circ s| = |l| \circ |s|$. **[1]**

5. Show that two connected graphs with the same number of vertices and edges are homotopy equivalent. **[4]**