Topological methods in combinatorics

Problem set 1 – Basics of general and algebraic topology

Published: **29. 02. 2024** - Hints: **14. 03. 2024** - Deadline: **21. 03. 2024**

Submit your solutions via the Postal Owl

1. Let X and Y be topological spaces, $f\colon X \to Y$ continuous function and $M, N \subset X$. Decide if the following claims hold, justify in each case.
   (a) If M is closed set, then $f(M)$ is closed. **[1]**
   Hint: Think of a subset of $\mathbb{R}$ that is not closed.
   (b) If M is open set, then $f(M)$ is open. **[1]**
   Hint: Think of a subset of $\mathbb{R}$ that is not open.
   (c) If M is connected then $f(M)$ is connected. **[1]**
   Hint: Verify that it satisfies the definition of connectivity.
   (d) If M is disconnected, then $f(M)$ is disconnected. **[1]**
   Hint: Think of a non-injective function

2. Let us consider the *deleted product* $X = (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x, x)\colon x \in \mathbb{R}^n\}$. (The topology on X is the subspace topology of the product topology.) Prove that X is homeomorphic to $S^{n-1} \times R^{n+1}$. **[4]**

   Hint: Realize that $\mathbb{R}^n \setminus \{0\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}$.

3. Prove that the two following definitions of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ coincide:
   (a) If $U \subseteq \mathbb{R}^n$ is open, then $f^{-1}(U)$ is also open.
   (b) For every $a \in \mathbb{R}^n$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $b \in \mathbb{R}^n$, if $\|b - a\| < \delta$, then $\|f(b) - f(a)\| \leq \varepsilon$.

   [2]

   Hint: Rewrite this using the definitions, and note that in $\mathbb{R}^n$, a set $X \subseteq \mathbb{R}^n$ is open iff every point $x \in X$ is contained within a ball $B(x, \varepsilon) \subseteq X$ for $\varepsilon > 0$.

4. Let $1 \leq r \leq n/2$, the Kneser graph $KG_{n,r}$ is the graph with vertex set all the r-subsets of $[n]$, i.e. $V(KG_{n,r}) = \{S \subset [n]\colon |S| = r\}$. A pair of r-sets S, T form an edge in $KG_{n,r}$ if $S \cap T = \emptyset$. Show that the chromatic number of $KG_{n,r}$ is at most $n - 2r + 2$. **[2]**

   Hint: Consider the coloring given by $c(F) = \min\{F, n - 2r + 2\}$.

5. (Gluing lemma for continuous functions) Let X be a topological space and $A_1, \ldots, A_n$ closed subspaces of X such that $X = \cup_{i=1}^n A_i$. Let $f\colon X \to Y$ function between topological spaces. Show that f is continuous if and only if the restriction of f to each A_i is continuous. **[2]**

   Hint: Verify that it satisfies the definition of continuous function with closed sets.

6. Show that the connected components of a topological space X are closed subsets. **[2]**

   Hint: Assume it is not the case. A connected component C is a maximal connected set. Consider the closure $Cl(C)$ of C. The $Cl(C)$ must be disconnected. Following this argument you will reach a contradiction.