Topological methods in combinatorics

Problem set 1 – Basics of general and algebraic topology

Published: **29. 02. 2024** - Hints: **14. 03. 2024** - Deadline: **21. 03. 2024**
Submit your solutions via the Postal Owl

1. Let X and Y be topological spaces, $f\colon X \to Y$ continuous function and $M, N \subset X$. Decide if the following claims hold, justify in each case.
   (a) If M is closed set, then $f(M)$ is closed. **[1]**
   (b) If M is open set, then $f(M)$ is open. **[1]**
   (c) If M is connected then $f(M)$ is connected. **[1]**
   (d) If M is disconnected, then $f(M)$ is disconnected. **[1]**
2. Let us consider the *deleted product* $X = (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x,x)\colon x \in \mathbb{R}^n\}$. (The topology on X is the subspace topology of the product topology.) Prove that X is homeomorphic to $S^{n-1} \times R^{n+1}$. **[4]**
3. Prove that the two following definitions of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ coincide:
   (a) If $U \subseteq \mathbb{R}^n$ is open, then $f^{-1}(U)$ is also open.
   (b) For every $a \in \mathbb{R}^n$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $b \in \mathbb{R}^n$, if $\|b - a\| < \delta$, then $\|f(b) - f(a)\| \leq \varepsilon$.

   [2]
4. Let $1 \leq r \leq n/2$, the Kneser graph $KG_{n,r}$ is the graph with vertex set all the r-subsets of $[n]$, i.e. $V(KG_{n,r}) = \{S \subset [n]\colon |S| = r\}$. A pair of r-sets S, T form an edge in $KG_{n,r}$ if $S \cap T = \emptyset$. Show that the chromatic number of $KG_{n,r}$ is at most $n - 2r + 2$. **[2]**
5. (Gluing lemma for continuous functions) Let X be a topological space and $A_1, \ldots, A_n$ closed subspaces of X such that $X = \cup_{i=1}^n A_i$. Let $f\colon X \to Y$ function between topological spaces. Show that f is continuous if and only if the restriction of f to each A_i is continuous. **[2]**
6. Show that the connected components of a topological space X are closed subsets. **[2]**