Topological methods in combinatorics

Class work 4 – The Brouwer fixed-point theorem and Sperner lemma

25. 04. 2024

Let $\Delta^n \subseteq \mathbb{R}^{n+1}$ be the standard n-dimensional simplex, that is the set

$$\left\{ \sum_{i=1}^{n+1} \alpha_i e_i : \alpha_i \geq 0, \sum_{i=1}^{n+1} \alpha_i = 1 \right\},$$

where the e_is are the standard unit vectors.

Theorem (Brouwer). *For every continuous function $f : \Delta^n \to \Delta^n$, there is a point $x \in \Delta^n$ satisfying $f(x) = x$.*

Definition. *A proper coloring of a triangulation of Δ^n is an assignment of $n+1$ colors to the vertices of the triangulation, so that the vertices receive all different colors, and points on each face F of Δ^n use only the colors that appear on the vertices of F.*

Theorem (Sperner's lemma). *Every properly colored triangulation of Δ^n contains an n-dimensional simplex whose vertices have all different colors, a* rainbow *simplex.*

1. Prove the Brouwer theorem from the Borsuk-Ulam theorem.
2. Prove the Sperner lemma using the Brouwer theorem.
3. Prove the Brouwer theorem using the Sperner lemma.
4. Prove the Sperner lemma.
   (a) Prove the case for $n = 1$ directly. (Can you say a bit more about the number of the rainbow simplicies?)
   (b) Use induction for $n \geq 2$.

Webpage: `https://iuuk.mff.cuni.cz/~chmel/2324/topmet/`