Topological methods in combinatorics

Class work 3 – The Borsuk-Ulam Theorem and its applications

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Definition 1. Let $\mathcal{F} \subseteq 2^X$ be a system of subsets of X.

The chromatic number $\chi(\mathcal{F})$ of $\mathcal{F}$ is the minimum m such that there exists a proper coloring $c\colon X \to \{1, \ldots, m\}$. That is, $|c(F)| > 1$ for each $F \in \mathcal{F}$.

Let the *m-colorability defect*, denoted by $cd_m(\mathcal{F})$, be the minimum size of a subset $Y \subseteq X$ such that the system of the sets of X that contain no points of Y can be properly colored by m colors.

Let $\mathrm{KG}(\mathcal{F})$ be a graph whose vertices are sets from $\mathcal{F}$ and whose vertices form an edge if and only the corresponding sets are disjoint.

Theorem 2 (Dolnikov's theorem)**.** *For any finite set system $\mathcal{F} \subseteq X$*

$$\chi(\mathrm{KG}(\mathcal{F})) \geq \mathrm{cd}_2(\mathcal{F}).$$

Definition 3. Let $\mathrm{SG}(n, k)$ denote *Schrijver graph* whose vertices are stable k-element subsets of $\{1, \ldots, n\}$. (Recall that $S \subseteq \{1, \ldots, n\}$ is *stable* if it does not contain any two adjacent elements modulo n. That is, if $i \in S$, then $i + 1 \notin S$, and if $n \in S$, then $1 \notin S$.)

1. Prove that if $f\colon X \to Y$ is a continuous mapping to a contractible space Y, then it is nullhomotopic.

2. Let $f\colon S^n \to S^n$ be a continuous mapping which is not surjective. Prove that f is nullhomotopic.

3. Let $f\colon S^n \to Y$ be a continuous mapping. Prove that the following statements are equivalent.

   (a) f is nullhomotopic.

   (b) f can be continuously extended to B^{n+1}.

4. For set systems $\mathcal{F}$ with $\chi(\mathrm{KG}(\mathcal{F})) \leq 2$ prove Dolnikov's theorem by direct combinatorial argument.

5. Show that every graph is $\mathrm{KG}(\mathcal{F})$ for some set system $\mathcal{F}$. Additionally, show that every graph is the intersection graph of some set system $\mathcal{F}$ (a set system induces an intersection graph in such a ways that the vertices are the sets from the set system, and two sets are joined by an edge iff their intersection is nonempty).

6. What is the number of vertices of $\mathrm{SG}(n, k)$?

7. Show that $\mathrm{SG}(n, k)$ is vertex-critical for chromatic number. In other words, if we remove an arbitrary vertex from $\mathrm{SG}(n, k)$, the chromatic number decreases.

Webpage: `https://iuuk.mff.cuni.cz/~chmel/2324/topmet/`