Topological methods in combinatorics

Class work 2 – Homotopy and simplicial complexes

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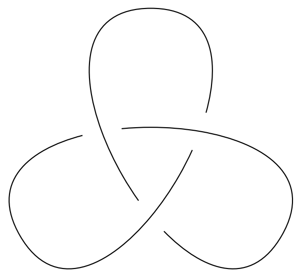
Lemma 1. *Let $f : X \to Y$ be a continuous function and $\simeq$ an equivalence on X satisfying that $x_1 \simeq x_2$ implies $f(x_1) = f(x_2)$. Then the function $\hat{f} : (X/\simeq) \to Y$ defined as $\hat{f}([x]) = f(x)$ is continuous.*

Lemma 2. *A continuous function $f : X \to Y$ from a compact space X to a Hausdorff space Y is closed.*

1. Show that the spaces $S^1 \times S^1$ and $I \times I/\{(x,0) \simeq (x,1), (0,y) \simeq (1,y)\}$ are homeomorphic.
2. Decide whether S^1 is homotopy equivalent to the trefoil:



3. Prove that X is contractible, i.e. homotopy equivalent to a point, if and only if for each Y is any continuous map $f : X \to Y$ nullhomotopic, i.e. homotopic to a constant map.
4. Show that the Möbius strip $M = I \times I/\{(x,0) \simeq (1-x,1)\}$ is homotopy equivalent to S^1.
5. Take a 2-dimensional sphere (in $\mathbb{R}^3$) and connect the north and south poles by a segment, obtaining a space X. Let Y be a 2-dimensional sphere with a circle attached by one point to the north pole of the sphere. Show that X and Y are homotopy equivalent.
6. Let K be an abstract 2-simplex and $\mathrm{sd}(K)$ its barycentric subdivision. Show that $|K|$ and $|\mathrm{sd}(K)|$ are homeomorphic.
7. Let σ be geometric 2-simplex, show that its diameter is attained at a pair of vertices of σ.

Useful definitions

- I denotes the unit interval $[0, 1] \subset \mathbb{R}$.
- Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two topological spaces. Their product $(X, \mathcal{O}_X) \times (Y, \mathcal{O}_Y)$ is a topological space with the universe $X \times Y$ and topology $\{U \times V : U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$.
- Let $(X, \mathcal{O})$ be a topological space, $\simeq$ an equivalence on X, and $q : X \to X/\simeq$ a function mapping x to its equivalence class $[x]$. Then $(X, \mathcal{O})/\simeq$ denotes the space with the universe $X/\simeq$ and topology $\{U \subset (X/\simeq) : q^{-1}(U) \in \mathcal{O}\}$.
- Continuous maps $f, g : X \to Y$ are homotopic, $f \sim g$, if there is a continuous map $F : X \times [0, 1] \to Y$ such that $F(\cdot, 0) = f(\cdot)$ and $F(\cdot, 1) = g(\cdot)$.
- Spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are homotopy equivalent, $X \simeq Y$, if there are continuous maps $f : X \to Y, g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.
- A simplex σ is a convex combination of a set $A \subset \mathbb{R}^n$ of affinely independent points. The dimension of σ is $\dim \sigma = |A| - 1$. That is, 2-simplex is a triangle.
- A family Δ of simplicies is a (geometric) simplicial complex if:

  (i) each face of any simplex $\sigma \in \Delta$ is also a simplex of Δ,

  (ii) the intersection $\sigma_1 \cap \sigma_2$ of $\sigma_1, \sigma_2 \in \Delta$ is a face of both σ_1 and σ_2.

  The union of all simplices in Δ is the polyhedron of Δ denoted by $|\Delta|$. The vertex set of Δ is the union of vertices of simplices in Δ.
- An abstract simplicial complex is a pair (V, K), where V is a set of vertices and K is a hereditary family of subsets of V, i.e. if $F \in K$ and $G \subseteq F$, then $G \in K$.
- Let K and L be abstract simplicial complexes. A mapping $f : V(K) \to V(L)$ is a simplicial mapping of K into L if for all $F \in K$ we have $f(K) \in L$.
- Let Δ_1 and Δ_2 be geometric simplicial complexes with K_1 and K_2 their abstract simplicial complexes. For a simplicial mapping $f : V(K_1) \to V(K_2)$, we define the mapping $|f| : |\Delta_1| \to |\Delta_2|$, an affine extension of f, as follows:

  (i) for x a vertex of Δ_1, $|f|$ maps x to the position of vertex the $f(x)$ in $|\Delta_2|$,

  (ii) otherwise, let $\sigma \in \Delta_1$ be the unique simplex with vertices $v_0, \ldots, v_k$ such that $x = \sum_{i=0}^{k} \alpha_i v_i$ with $\alpha_i > 0$ and $\sum_{i=0}^{k} \alpha_i = 1$, then we put $|f|(x) = \sum_{i=0}^{k} \alpha_i \cdot |f|(v_i)$.
- For an abstract simplicial complex K, we define it barycentric subdivision $\mathrm{sd}(K)$ to be a simplicial complex whose vertices are faces of K. A subset of faces of K is a simplex of $\mathrm{sd}(K)$ if they form a chain ordered by inclusion.
- The diameter of a geometric simplex σ is the maximum distance between any two points of σ.
