Topological methods in combinatorics

Class work 1 – Basics of general and algebraic topology

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1. Let $f\colon (X,\mathcal{O}_X) \to (Y,\mathcal{O}_Y)$ and $g\colon (Y,\mathcal{O}_Y) \to (Z,\mathcal{O}_Z)$ be a pair of continuous functions. Define $h\colon (X,\mathcal{O}_X) \to (Z,\mathcal{O}_Z)$ by setting $h(x) = g(f(x))$. Show that h defines a continuous function.

2. Show that an $f : X \to Y$ is continuous if and only if $\forall A \subseteq Y$, if A is closed, then $f^{-1}(A)$ is closed.

3. Construct an $f : X \to Y$ that is open (i.e., the image of an open set is an open set) but not continuous.

4. Decide and justify if the following functions are continuous
   (a) $id\colon x \in (\mathbb{R},\mathcal{O}_{\text{Euclid}}) \mapsto x \in (\mathbb{R},\mathcal{O}_{\text{Discrete}})$.
   (b) $id\colon x \in (\mathbb{R},\mathcal{O}_{\text{Discrete}}) \mapsto x \in (\mathbb{R},\mathcal{O}_{\text{Euclid}})$.
   (c) Let $p(x_0,\dots,x_n)$ be a polynomial on variables $x_0,\dots,x_n$ with real coefficients, define $eval_p\colon x \in (S^n,\mathcal{O}_{\text{Euclid}}) \mapsto p(x) \in (\mathbb{R},\mathcal{O}_{\text{Euclid}})$.

5. Show that $f\colon X \to Y$ is a homeomorphism if and only if f is continuous, bijective and open/closed.

6. (Product topology) Let $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ be topological spaces. The cartesian product $X \times Y$ has the collection $\{U \times V\colon U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$ as basis. Decide if $(\mathbb{R}^2,\mathcal{O}_{Euclid})$ and $\mathbb{R}\times\mathbb{R}$ with the product topology are homeomorphic.

7. (Quotient topology) Let $(X,\mathcal{O}_X)$ topological space and $\simeq$ an equivalence relation on the elements of X. The topology on $X/\simeq$ on the equivalence classes is given by $q\colon X \to X/\simeq$, $U \subset X/\simeq$ is open if and only if $q^{-1}(U)$ is open. Show that $[0,1]/\{0,1\}$ and S^1 are homeomorphic.

8. Show that $\mathbb{R}$, $S^1 \setminus \{(0,1)\}$ and $(0,1)$ are homeomorphic.

9. Show that S^1 and $\partial([0,1]^2)$ are homeomorphic.

10. Prove that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \in X \times X\colon x \in X\}$ is closed in $X \times X$.

11. Let $f,g : X \to Y$ be continuous. If Y is Hausdorff, then $\{x \in X : f(x) = g(x)\}$ is closed in X.

Useful definitions

- Topological space: $(X, \mathcal{O}) : \mathcal{O} \subseteq \mathcal{P}(X), \emptyset \in \mathcal{O}, X \in \mathcal{O}$, and $\mathcal{O}$ is closed under arbitrary union and finite intersection.
- For a metric space (X, d), where $d : X \times X \to \mathbb{R}_0^+$ is a metric, we can define a topology $\mathcal{O}_d$ by taking all open balls $B(x, \varepsilon) = \{y \in X : d(x, y) \leq \varepsilon\}$, and then closing the family under union. (In other words, we take the family of all open balls as the basis of the topology.)
- For a topology $\mathcal{O}$, its basis is $\mathcal{B} \subseteq \mathcal{O}$ such that every $U \in \mathcal{O}$ can be written as a union of members of $\mathcal{B}$.
- The discrete topology on X is $\mathcal{O}_{\text{Discrete}} = \mathcal{P}(X)$.
- A function $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is continuous if $\forall A \in \mathcal{O}_Y,\ f^{-1}(A) \in \mathcal{O}_X$.
- A bijection $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a homeomorphism if f and f^{-1} are both continuous.
- The n-sphere is $S^n := \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$.
- A topological space $(X, \mathcal{O})$ is Hausdorff, if for all $x \neq y \in X : \exists U_x, U_y \in \mathcal{O}$ that are disjoint (and open) such that $x \in U_x, x \notin U_y$, and $y \in U_y, y \notin U_x$.
- A topological space X is connected if it can't be written as a union of two disjoint open sets.
- Connected components of X are inclusion-wise maximal subsets that are connected (as subspaces of X).