OPTIMIZATION METHODS: CLASS 9

Complementary slackness

D(Slack**):** Suppose we have a system of linear inequalities (S) and, more specifically, the j-th inequality

$$a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + \ldots + a_{jn}x_n \leq b_j.$$

Suppose we are also given a vector x' that satisfies the j-th inequality. Then the *slack* of the j-th inequality and the solution x' is $s_j^{(S)} = b_j - \sum_{i=1}^n a_{ji}x'_i$.

Notice that it always holds that $s_j^{(S)} \geq 0$. If the inequality is $\geq$, we define the slack as $s_j^{(S)} = \sum_{i=1}^n a_{ji}x'_i - b_j$, so that again $s_j^{(S)} \geq 0$.

T(Complementary slackness**):** Assume we have a linear program (P) and its dual (D) of the following form.

$$\max c^T x, Ax \leq b, x \geq 0, \tag{P}$$
$$\min b^T y, A^T y \geq c, y \geq 0. \tag{D}$$

We are also given a pair of feasible solutions of the primal and dual (x', y'). Then the following holds: The pair (x', y') is a pair of optimal solutions if and only if all the following conditions are satisfied:

$$\forall i \in \{1, \ldots, n\} \colon x'_i \cdot s_i^{(D)} = 0, \tag{1}$$
$$\forall j \in \{1, \ldots, m\} \colon s_j^{(P)} \cdot y'_j = 0. \tag{2}$$

Exercise one Find an integral polytope $\{x;\ Ax \leq b, x \geq 0\}$, where A is a matrix of size at least 3×3 and both A and b have integer values, but A is not totally unimodular. Can you give an example where A contains only -1, 0 a 1? Can you give one even if we forbid the value -1?

Exercise two During an optimization exam, Josef K. copied from his neighbor the statement of a dual program and a feasible solution of the primal:

The dual is:

$$\begin{aligned}
\max 2x_1 + 3x_2 + 5x_3 + 4x_4& \\
x_1 + 2x_2 + 3x_3 + x_4 &\leq 5 \\
x_1 + x_2 + 2x_3 + 3x_4 &= 3 \\
x_1 + x_2 + x_3 + x_4 &\geq 1 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}$$

The feasible primal solution is $y = (4, 0, 0)$. However, the goal of the exercise was to check if y is an optimum or not. Use complementary slackness to check this. (You do not even need to dualize.)

Exercise three Josefka K. also cheated during her exam, but she copied a statement of the primal and an optimal solution of the dual:

$$
\begin{aligned}
\min 10x_1 - 4x_2 \\
x_1 + 0.6x_3 + 4x_4 &\geq 43 \\
x_1 - x_2 + 0.6x_3 + 10x_4 &\geq 27 \\
x_1 - x_2 - 0.4x_3 - x_4 &\geq 24 \\
x_1 - x_2 - 0.4x_3 - 2x_4 &\geq 22 \\
x_1 + 3.6x_3 - 3x_4 &\geq 56 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

The optimal dual solution is $y = (3.36, 0, 0, 6.48, 0.16)$. However, the goal was to compute the optimal solution of the primal. Help Josefka using complementary slackness.

EXERCISE FOUR Formulate the complementary slackness theorem if the primal LP is of the standard simplex form:

$$
\begin{aligned}
\max c^T x \\
Ax &= b \\
x &\geq 0
\end{aligned}
$$

(What is the form of the dual?)

EXERCISE FIVE For the LP and the dual from the third exercise find a pair of vectors x a y such that:

$$
\forall i \in \{1, \dots, n\} \colon x_i \cdot s_i^{(D)} = 0, \tag{1}
$$

$$
\forall j \in \{1, \dots, m\} \colon s_j^{(P)} \cdot y_j = 0. \tag{2}
$$

but x and y **are not** a pair of optimal solutions.

Tip: Find the difference between the statement of the theorem and this exercise.