OPTIMIZATION METHODS: CLASS 8

Total unimodularity

D: A square matrix $M \in \mathbb{R}^{n \times n}$ is *unimodular* if its determinant is $-1, 0$ or 1.

D: An arbitrary matrix $M \in \mathbb{R}^{m \times n}$ is *totally unimodular* if the determinant of every square submatrix is equal to $-1, 0$ or 1.

D: We call a polytope *integral* if all its vertices have integer coefficients.

T(Cramer rule): Consider a system of equations $Ax = b$ where A has nonzero determinant. A solution to such a system can be expressed as the quotient of determinants:

$$x_i = \frac{\det(A_i)}{\det(A)}.$$

The matrix A_i is formed by taking the matrix A and replacing the i-th column with the vector b.

T:For a given linear program $\max c^T x, Ax \leq b, x \geq 0$, suppose that b is an integral vector and A is a totally unimodular matrix. Then the polytope $Ax \leq b, x \geq 0$ is integral.

T(Corollary of the previous theorem): Consider an integer program $ILP = \max c^T x, Ax \leq b, x \geq 0, x \in \mathbb{Z}$. Its linear relaxation is the linear program $LP = \max c^T x, Ax \leq b, x \geq 0$.

Suppose that b is an integral vector and A is a totally unimodular matrix. Then any vertex optimal solution of LP is an optimal solution of ILP.

Exercise one

Let A be a totally unimodular matrix.

- Show that A contains only entries with values $-1, 0, 1$.
- Show that A^T, $\binom{A}{-A}$ a $(A|I)$ are totally unimodular matrices.

Exercise two

Check if the following matrix is totally unimodular (without use of the following exercise):

$$\begin{matrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{matrix}$$

Exercise three

Consider a matrix A of size $m \times n$ whose *rows* can be partitioned into two disjoint groups B and C. Suppose that the following holds:

1. $A \in \{-1, 0, 1\}^{m \times n}$.
2. Every column has at most two non-zero values,
3. If the two non-zero values in a column have *the same* sign, then one corresponding row with a non-zero value belongs to B and the other row belongs to C.
4. If the two non-zero values in a column have *opposite* sign, then both the corresponding rows belong to B or both belong to C.

As an example, the matrix from the previous exercise has such a partition.

Prove that with such a partition, A must be totally unimodular.

Exercise four

Prove that the incidence matrix of a directed graph is always totally unimodular.

Exercise five

Prove that for an *undirected* graph, its incidence matrix is totally unimodular if and only if the graph is bipartite. Explain how this claim can be used to prove Konig's theorem.

EXERCISE SIX Find an integral polytope $\{x;\ Ax \leq b, x \geq 0\}$, where A is a matrix of size at least 3×3 and both A and b have integer values, but A is not totally unimodular. Can you give an example where A contains only -1, 0 a 1? Can you give one even if we forbid the value -1?

EXERCISE SEVEN Imagine one column vector v with values only $\{0, 1\}$. We say that v has *interval form* if v has all the 1-values in one continuous interval (possibly of length zero). A matrix M has *interval form* if all its columns have interval form.

Show that every matrix M with interval form is also totally unimodular.