OPTIMIZATION METHODS: CLASS 7

Duality

D: Assume we have a linear program with n variables and m constraints:

$$\max c^T x, Ax \leq b, x \geq 0.$$

Then its *dual* is a linear program with m variables and n constraints:

$$\min b^T y, A^T y \geq c, y \geq 0.$$

T(Weak duality**):** Given a maximization linear program $\max c^T x$ and a dual minimization program $\min b^T y$. Then for any feasible solution x and any dual feasible solution y, we have that $c^T x \leq b^T y$. In other words, the value of the dual solution is an upper bound on the value of any feasible primal solution.

Original program:	*In the dual:*
maximum	minimum
$\max c^T x$	$\min b^T y$
m constraints n variables	m variables n constraints
the i-th constraint is $\leq$ the i-the constraint is $\geq$ the i-th constraint is $=$	$y_i \geq 0$ $y_i \leq 0$ $y_i \in \mathbb{R}$
$x_j \geq 0$ $x_j \leq 0$ $x_j \in \mathbb{R}$	the j-th constraint is $\geq$ the j-th constraint is $\leq$ the j-th constraint is $=$

Exercise one

Dualize the following LP:

$$\begin{aligned} \max x_1 - 2x_2 + 3x_4 \\ x_2 \leq 0 \\ x_4 \geq 0 \\ x_2 - 6x_3 + x_4 \leq 4 \\ -x_1 + 3x_2 - 3x_3 = 0 \\ 6x_1 - 2x_2 + 2x_3 - 4x_4 \geq 5 \end{aligned}$$

Exercise two

Dualize the linear programming relaxation of the integer program for Minimum vertex cover for a weighted graph $G = (V, E, w)$. To be precise, the task is to dualize the following:

$$\begin{aligned} \min \sum_{v \in V} w(v) x_v \\ \forall e = (uv) \in E : x_u + x_v \geq 1 \\ \forall v \in V : x_v \geq 0 \end{aligned}$$

EXERCISE THREE Formulate a linear program which solves the task SHORTEST s, t-PATH in an unweighted undirected graph. (We have seen this LP already at the second practical.)

Explain your idea behind the LP, and then dualize it.

Followup question: Does the dual LP also have some "idea" behind it?

EXERCISE FOUR We are given some linear program (P) which has some optimum solution, but we do not know it yet. The program (P) is of this form:

$$\max c^T x, Ax \leq b, x \geq 0.$$

Using duality, formulate a new LP that satisfies the following:

- it has no objective function (so it is just a polytope),
- if somebody gives us any feasible solution x of the polytope, we can read from its coordinates the optimum solution of the program (P).

EXERCISE FIVE Prove that a dual of a dual of a linear program L is the same program as L.

EXERCISE SIX Prove or disprove the following:

If a linear program has an optimum solution with integral coefficients, then its dual also has an optimum solution with integral coefficients.