OPTIMIZATION METHODS: CLASS 6

The simplex algorithm, counting faces

EXERCISE ONE Apply the simplex method on the following LPs. At some point it may not be possible to continue. Try to draw the polytope P and reason why the algorithm stopped. Is the issue dependent on the value function, or just on the polytope?

- Optimize the function $\max 3x_1 + x_2$ on the polytope P:

$$\begin{array}{rcrcr} x_1 & - & x_2 & \leq & -1 \\ -x_1 & - & x_2 & \leq & -3 \\ 2x_1 & - & x_2 & \leq & 2 \\ \multicolumn{5}{l}{x_1, x_2 \geq 0.} \end{array}$$

- Optimize the function $\max 4x + 5y + 3z$ on the polytope P:

$$\begin{aligned} x + y + 2z &\geq 20 \\ 5x + 6y + 5z &\leq 50 \\ x + 3y + 5z &\leq 30 \\ x, y, z &\geq 0 \end{aligned}$$

D: Let P be some convex polytope in $\mathbb{R}^d$. We say that a hyperplane H is a *supporting hyperplane* if it does not cut the polytope.

In other words, if the hyperplane H is defined as $\{x \in \mathbb{R}^d | c^T x = t\}$, then we say H is supporting if and only if it holds that $\forall y \in P : \{c^T y \leq t\}$ or it holds that $\forall y \in P : \{c^T y \geq t\}$.

D: A *face* F of a polytope P is any set of the form $F = P \cap H$ for any supporting hyperplane H.

Note that our definition allows that $P \cap H = \emptyset$. We also count P itself as a face. These two faces P,$\emptyset$ are *improper* faces, the rest of the faces (whenever $P \cap H \neq \emptyset$) are called *proper*.

D: A *vertex* of a polytope P is a face of dimension 0 (a single point). An *edge* is any face of dimension 1 (a line segment, half-line or a line). On the other side of the spectrum, a *facet* of P is a face of dimension $d - 1$.

D: A d-dimensional *simplex* is a polytope which arises as a convex hull of any $d + 1$ affinely independent points. All simplexes are structurally the same, so whenever you think about a simplex, you can consider the set:

$$\mathrm{conv}(0, (0, 0, \ldots, 0, 1), (0, 0, \ldots, 1, 0), \ldots, (1, 0, \ldots, 0, 0))$$

D: A d-dimensional *cube* is a convex hull of all binary vectors in $\mathbb{R}^d$.

D: A d-dimensional *crosspolytope* is a convex hull of all points $\pm e_i$ (for $i \in 1, \ldots, d$). Here, e_i is a vector with just a single 1 on the i-th coordinate and 0 otherwise.

Example: In three dimensions, a crosspolytope is a convex hull of $(0, 0, 1)$, $(0, 0, -1)$, $(0, 1, 0)$, $(0, -1, 0)$, $(1, 0, 0)$ and $(-1, 0, 0)$.)

T(Vertex description of a polytope**):** Every bounded convex polytope is equal to the convex hull of all its vertices. Bounded polytopes therefore can be described using all their halfspaces (then the polytope is their intersection) or their vertices (then the polytope is their convex hull).

T(Basic solutions are exactly vertices of a polytope**):** A point x_i is a vertex of a convex polytope $Ax \leq b$ defined in $\mathbb{R}^d$ if and only if x_i is a *basic solution* of $Ax \leq b$.

In other words: In order for x_i to be a vertex, it has to be inside the polytope (satisfy all the inequalities) and it also must satisfy d *linearly-independent* inequalities with equality.

Exercise two Prove or disprove the following: Consider a d-dimensional simplex $S \subseteq \mathbb{R}^d$. Is there a *cutting* hyperplane H such that neither $S \cap H^+$ nor $S \cap H^-$ is a simplex itself?

To be precise, by H^+ and H^- we mean the closed halfspaces defined by the hyperplane H.

Exercise three Compute the number of k-dimensional faces of a d-dimensional simplex.

Hint: Calculate the numbers for the $2D$, $3D$ simplex first, then come up with a general formula.

Exercise four Calculate how many halfspaces are needed to define a crosspolytope of dimension d. (This number is equal the number of facets of the crosspolytope.)

Exercise five

1. Recall that a face of a simplex is again a simplex.
2. Show that a face of a cube is again a (shifted) cube.
3. Does it hold that a facet of a crosspolytope is again a (shifted) crosspolytope? If not, what is it?