OPTIMIZATION METHODS: CLASS 5

The simplex algorithm

EXERCISE ONE Apply "brute force" to find all vertices of a polytope defined as follows:

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 14 \\ 2x_1 + 5x_2 + 5x_3 &\leq 30 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \end{aligned}$$

By brute force we mean checking all the d-tuples of inequalities.

EXERCISE TWO Transform the polytope into the standard equational form:

$$\begin{aligned} x_1 + x_2 &\leq 3 \\ x_2 + x_3 &\leq 12 \\ x_1 + 3x_2 - x_4 &\geq -7 \\ x_5 &\geq 6 \\ x_2 + x_5 &\leq 13 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Additionally, find some basic feasible solution for this equational form. Try to find a good solution of the first task to help you with the second task.

EXERCISE THREE Suppose that we are given the following problem:

$$\begin{aligned} \max x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 & \\ x_1 - x_5 + x_6 &= 20 \\ x_1 + x_3 + x_7 &= 30 \\ x_1 + x_2 + x_4 + x_8 &= 10 \\ x_2 - x_3 - x_4 + x_5 + x_9 &= 1 \\ x_1, x_2, \ldots, x_9 &\geq 0 \end{aligned}$$

and an initial basic solution $(0, 0, 0, 0, 0, 20, 30, 10, 1)$. Execute one step of the simplex algorithm. Which variable did you pick for your step and why?

EXERCISE FOUR Solve the following LP completely, doing all the steps:

$$\begin{aligned} \max 2x_1 - x_2 + 2x_3 \text{ subject to:} & \\ 2x_1 + x_2 &\leq 10 \\ x_1 + 2x_2 - 2x_3 &\leq 20 \\ x_2 + 2x_3 &\leq 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Exercise five

Find an initial basic solution of the following problem by "guessing right":

$$\begin{aligned} \max 4x_2 - x_4 & \\ 3x_1 + x_2 - 2x_4 &= 5 \\ -x_2 + x_3 &= -2 \\ -2x_1 + 8x_2 + x_3 &= 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Then, try to find one initial solution using the simplex method and discuss which process was faster.

Exercise six

Apply the simplex method on the following LPs. At some point it may not be possible to continue. Try to draw the polytope P and reason why the algorithm stopped. Is the issue dependent on the value function, or just on the polytope?

- Optimize the function $\max 3x_1 + x_2$ on the polytope P:

$$\begin{array}{rcrcr} x_1 & - & x_2 & \leq & -1 \\ -x_1 & - & x_2 & \leq & -3 \\ 2x_1 & - & x_2 & \leq & 2 \\ x_1, x_2 \geq 0. & & & & \end{array}$$

- Optimize the function $\max 4x + 5y + 3z$ on the polytope P:

$$\begin{aligned} x + y + 2z &\geq 20 \\ 15x + 6y + 5z &\leq 50 \\ x + 3y + 5z &\leq 30 \\ x, y, z &\geq 0 \end{aligned}$$